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MATHEMATICS

On Direct Decompositions of Torsionfree Abelian Groups

by

L. JEŚMANOWICZ

Presented by A. MOSTOWSKI on May 25, 1960

In 1958 L. Fuchs [3] suggested the following problem: Let r_1, r_2, s_1, s_2 be integers such that $r_1 + r_2 = s_1 + s_2$ and $r_1 \neq s_1, r_1 \neq s_2$. Do such indecomposable torsionfree groups A_1, A_2, B_1, B_2 exist that $A_1 + A_2 = B_1 + B_2$, where the rank of A_i is r_i and that of B_i is s_i $(i = 1,2)$ *) ?

In this note we give an affirmative solution to this problem and a solution of the analogous problem concerning decompositions of the group into three direct summands.

THEOREM 1. *For any two systems of integers (r_1, r_2) and (s_1, s_2) such that $1 \leqslant r_1 \leqslant r_2$, $r_1 < s_1 \leqslant s_2$, $r_1 + r_2 = s_1 + s_2$, there exist indecomposable torsionfree Abelian groups A_1, A_2, B_1, B_2 of the ranks r_1, r_2, s_1, s_2 respectively, such that*

$$A_1 + A_2 = B_1 + B_2 .$$

Proof. Let $(x_1, x_2, ..., x_{r_1}, y_1, y_2, ..., y_{r_2})$ be a basis for $(r_1 + r_2)$-dimensional vector space V over the field of rational numbers and let

$$x_i' = 3x_i - y_i,\ y_i' = 2x_i - y_i \quad (i = 1, 2, ..., r_1).$$

Then $(x_1', x_2', ..., x_{r_1}', y_1', y_2', ..., y_{r_1}', y_{r_1+1}, ..., y_{r_2})$ is also a basis for V and

$$x_i = x_i' - y_i',\ y_i = 2x_i' - 3y_i' \quad (i = 1, 2, ..., r_1).$$

Since $s_1 - r_1 > 0$, $r_2 - s_1 = s_2 - r_1 > 0$ and $r_2 - r_1 = (s_1 - r_1) + (r_2 - s_1)$, we can split the set $(y_{r_1+1}, ..., y_{r_2})$ into two non empty subsets $(y_{r_1+1}, ..., y_{s_1+1})$ and $(y_{s_1+1}, ..., y_{r_2})$.

For the sake of brevity let us denote

$$X = x_1 + x_2 + ... + x_{r_1}, \qquad Y = y_1 + y_2 + ... + y_{r_1},$$

$$X' = x_1' + x_2' + ... + x_{r_1}', \qquad Y' = y_1' + y_2' + ... + y_{r_1}',$$

$$Y_1 = y_{r_1+1} + ... + y_{s_1}, \qquad Y_2 = y_{s_1+1} + ... + y_{r_2} .$$

*) For special cases of the problem we refer to R. Baer [1] and B. Jonsson [2].

Consider the following subgroups of V under vector addition:

$$A_1 = \left\{ \frac{x_1}{p_1^n}, \frac{x_2}{p_2^n}, \dots, \frac{x_{r_1}}{p_{r_1}^n}, \frac{X}{p_0^n} \right\};$$

$$A_2 = \left\{ \frac{y_1}{p_1^n}, \frac{y_2}{p_2^n}, \dots, \frac{y_{r_1}}{p_{r_1}^n}, \frac{y_{r_1+1}}{p_{r_1+1}^n}, \dots, \frac{y_{r_2}}{p_{r_2}^n}, \frac{Y}{p_0^n}, \frac{Y+Y_1}{3}, \frac{Y+Y_2}{2} \right\},$$

$$B_1 = \left\{ \frac{x_1'}{p_1^n}, \frac{x_2'}{p_2^n}, \dots, \frac{x_{r_1}'}{p_{r_1}^n}, \frac{y_{r_1+1}}{p_{r_1+1}^n}, \dots, \frac{y_{s_1}}{p_{s_1}^n}, \frac{X'}{p_0^n}, \frac{X'-Y_1}{3} \right\},$$

$$B_2 = \left\{ \frac{y_1'}{p_1^n}, \frac{y_2'}{p_2^n}, \dots, \frac{y_{r_1}'}{p_{r_1}^n}, \frac{y_{s_1+1}}{p_{r_1+1}^n}, \dots, \frac{y_{r_2}}{p_{r_2}^n}, \frac{Y'}{p_0^n}, \frac{Y'-Y_2}{2} \right\},$$

where $p_0, p_1, \dots, p_{r_2}$ are different primes greater than 3 and n are arbitrary integers.

Evidently, the ranks of A_1, A_2, B_1, B_2 are r_1, r_2, s_1, s_2 respectively.

Since A_1 is contained in the vector space spanned by $(x_1, x_2, \dots, x_{r_1})$ and A_2 is contained in the vector space spanned by $(y_1, y_2, \dots, y_{r_2})$, we see that A_1 and A_2 have only 0 in common, so that $A_1 + A_2$ is a direct sum of A_1 and A_2. For similar reasons $B_1 + B_2$ is a direct sum of B_1 and B_2.

We omit the proof that the groups $A_1 + A_2$ and $B_1 + B_2$ are equal. It can be proved similarly as in Theorem 2, only it is much less complicated.

In virtue of the theorem of de Groot [4] the group A_1 is indecomposable. The group A_2 is also indecomposable, which follows from the below reasoning due to B. Jonsson [5]. If $A_2 = A_2' + A_2''$, then there exist homomorphisms φ and ψ, which map A_2 onto A_2' and A_2'', respectively, in such a way that $\varphi(a) + \psi(a) = a$ for every $a \in A_2$, $\varphi(a) = a$, $\psi(a) = 0$ for every $a \in A_2'$ and $\varphi(a) = 0, \psi(a) = a$ for every $a \in A_2''$.

Consider $y_j = \varphi(y_j) + \psi(y_j)$, $j = 1, 2, \dots, r_2$.

Since $\varphi(y_j) = p_j^m \varphi(y_j/p_j^m)$ for $m = 1, 2, 3, \dots$, we infer that the element $\varphi(y_j)$, which is divisible in A_2 by p_j^m for all $m = 1, 2, 3, \dots$, belongs to the group $\{y_j/p_j^n, n = 1, 2, 3 \dots\}$. For similar reasons $\psi(y_j) \in \{y_j/p_j^n, n = 1, 2, 3 \dots\}$.

But the groups $\{y_j/p_j^n; n = 1, 2, 3, \dots\}$ are indecomposable, so that $\varphi(y_j) = 0$ or $\psi(y_j) = 0$. Suppose $\psi(y_j) = 0$ for $j = 1, \dots, k < r_1$ and $\varphi(y_j) = 0$ for $j = k+1, \dots, r_1$. Then*)

$$\varphi\left(\frac{y_1 + y_2 + \dots + y_{r_1}}{p_0}\right) = \frac{1}{p_0} \varphi(y_1 + y_2 + \dots + y_{r_1}) =$$

$$= \frac{\varphi(y_1) + \varphi(y_2) + \dots + \varphi(y_k)}{p_0} = \frac{y_1 + y_2 + \dots + y_k}{p_0}$$

which is impossible, because $p_0^{-1}(y_1 + y_2 + \dots + y_k) \in A_2$. Thus we must have $\varphi(y_j) = 0$ for $j = 1, 2, \dots, r_1$ or $\psi(y_j) = 0$ for $j = 1, 2, \dots, r_1$.

*) If $a \in A_2$ and $a/s \in A_2$ then $\varphi(a/s)$ and $\psi(a/s)$ are solutions of the equations $sx = \varphi(a)$, $sy = \psi(a)$ respectively. Since the group A_2 is torsionfree, these solutions are unique and therefore $\varphi(a/s) = \varphi(a)/s$, $\psi(a/s) = \varphi(a)/s$.

Let $\psi(y_j)=0$ for $j=1,2,\dots,r_1$. If $\psi(y_j)=0$ for $j=r_1+1,\dots,k<s_1$ and $\varphi(y_j)=0$ for $j=k+1,\dots,s_1$, then $\varphi\left(\frac{Y+Y_1}{3}\right)=\frac{1}{3}\varphi(Y+Y_1)=$ $=\frac{\varphi(Y)+\varphi(y_{r_1+1})+\cdots+\varphi(y_{s_1})}{3}=\frac{Y+y_{r_1+1}+\cdots+y_k}{3}$, which is impossible because $3^{-1}(Y+y_{r_1+1}+\cdots+y_k)\,\bar{\epsilon}\,A_2$. We must therefore have $\psi(y_j)=0$ $j=r_1+1,\dots,s_1$.

If $\psi(y_i)=0$ for $j=s_1+1,\dots,k<r_2$ and $\varphi(y_i)=0$ for $j=k+1,\dots,r_2$, then $\varphi\left(\frac{Y+Y_2}{2}\right)=\frac{1}{2}(Y+y_{s_1+1}+\cdots+y_k,$ which is impossible because $2^{-1}(Y+y_{s_1+1}+\cdots+y_k)\,\bar{\epsilon}\,A_2$. Thus we have $\psi(y_i)=0$ for $j=s_1+1,\dots,r_2$. Since any element of A_2 is a linear combination of $y_1,y_2,\dots,y_{r_2}$ with rational coefficients, it follows that $\psi(a)=0$ for any $a\in A_2$. We therefore conclude that $A_2''=\{0\}$ and $A_2'=A_2$. Thus A_2 is indecomposable.

Similar arguments show that the groups B_1 and B_2 are indecomposable and the proof of the Theorem is complete.

THEOREM 2. *For any two systems of integers (r_1,r_2,r_3) and (s_1,s_2,s_3) such that*

$$(1)\qquad 1\leqslant r_1\leqslant r_2\leqslant r_3,\quad r_1<s_1\leqslant s_2\leqslant s_3,\quad r_1+r_2+r_3=s_1+s_2+s_3,$$

there exist indecomposable torsionfree Abelian groups A_i, B_i $(i=1,2,3)$ of the ranks r_i, s_i, respectively, such that $A_1+A_2+A_3=B_1+B_2+B_3$.

Proof. Let $(x_1,x_2,\dots,x_{r_1},y_1,y_2,\dots,y_{r_2}\,z_1,z_2,\dots,z_{r_3})$ be a basis for the $(r_1+r_2+r_3)$-dimensional vector space V over the field of rational numbers and let

$$x_i'=a_{11}x_i+a_{12}y_i+a_{13}z_i$$

$$y_i'=a_{21}x_i+a_{22}y_i+a_{23}z_i,\qquad (i=1,2,\dots,r_1)$$

$$z_i'=a_{31}x_i+a_{32}y_i+a_{33}z_i,$$

where a_{ik} are integers and $|\det(a_{ik})|=1$. Then

$$(x_1',x_2',\dots,x_{r_1}',y_1',y_2',\dots,y_{r_1}',y_{r_1+1},\dots,y_{r_2},z_1',z_2',\dots,z_{r_1}',z_{r_1+1},\dots,z_{r_3})$$

is also a basis for V.

According to (1) there exist non-negative integers u_i,v_i $(i=1,2,3)$ such that:

$$r_1=r_1,\quad r_2=r_1+u_1+u_2+u_3,\quad r_3=r_1+v_1+v_2+v_3,$$

$$s_1=r_1+u_1+v_1,\quad s_2=r_1+u_2+v_2,\quad s_3=r_1+u_3+v_3.$$

We give the proof only in the case if $u_i>0$ and $v_i>0$ for $i=1,2,3$. When some u_i or v_i are zeros the proof is much less complicated.

We denote:

$$X = x_1 + x_2 + \dots + x_{r_1}, \quad Y = y_1 + y_2 + \dots + y_{r_1}, \quad Z = z_1 + z_2 + \dots + z_{r_1},$$

$$X' = x_1' + x_2' + \dots + x_{r_1}', \quad Y' = y_1' + y_2' + \dots + y_{r_1}', \quad Z' = z_1' + z_2' + \dots + z_{r_1}',$$

$$Y_1 = y_{r_1+1} + \dots + y_{r_1+u_1}, \quad Y_2 = y_{r_1+u_1+1} + \dots + y_{r_1+u_1+u_2},$$

$$Y_3 = y_{r_1+u_1+u_2+1} + \dots + y_{r_2},$$

$$Z_1' = z_{r_1+1} + \dots + z_{r_1+v_1}, \quad Z_2 = z_{r_1+v_1+1} + \dots + z_{r_1+v_1+v_2},$$

$$Z_3 = z_{r_1+v_1+v_2+1} + \dots + z_{r_3},$$

Consider the following subgroups of V under vector addition (everywhere n is assumed to run over the integers $1, 2, \dots$).

$$A_1 = \left\{ \frac{x_1}{p_1^n}, \frac{x_2}{p_2^n}, \dots, \frac{x_{r_1}}{p_{r_1}^n}, \frac{X}{p_0^n} \right\},$$

$$A_2 = \left\{ \frac{y_1}{p_1^n}, \frac{y_2}{p_2^n}, \dots, \frac{y_{r_1}}{p_{r_1}^n}, \frac{y_{r_1+1}}{p_{r_1+1}^n}, \dots, \frac{y_{r_2}}{p_{r_2}^n}, \frac{Y}{p_0^n}, \frac{Y+Y_1}{2}, \frac{Y+Y_2}{3}, \frac{Y+Y_3}{5} \right\},$$

$$A_3 = \left\{ \frac{z_1}{p_1^n}, \frac{z_2}{p_2^n}, \dots, \frac{z_{r_1}}{p_{r_1}^n}, \frac{z_{r_1+1}}{q_{r_1+1}^n}, \dots, \frac{z_{r_3}}{q_{r_3}^n}, \frac{Z}{p_0^n}, \frac{Z+Z_1}{5}, \frac{Z+Z_2}{2}, \frac{Z+Z_3}{3} \right\},$$

$$B_1 = \left\{ \frac{x_1'}{p_1^n}, \frac{x_2'}{p_2^n}, \dots, \frac{x_{r_1}'}{p_{r_1}^n}, \frac{y_{r_1+1}}{p_{r_1+1}^n}, \dots, \frac{y_{r_1+u_1}}{p_{r_1+u_1}^n}, \frac{z_{r_1+1}}{q_{r_1+1}^n}, \dots, \frac{z_{r_1+v_1}}{q_{r_1+v_1}^n}, \frac{X'}{p_0^n}, \frac{X'-Y_1}{2}, \frac{X'-Z_1}{5} \right\}$$

$$B_2 = \left\{ \frac{y_1'}{p_1^n}, \frac{y_2'}{p_2^n}, \dots, \frac{y_{r_1}'}{p_{r_1}^n}, \frac{y_{r_1+u_1+1}}{p_{r_1+u_1+1}^n}, \dots, \frac{y_{r_1+u_1+u_2}}{p_{r_1+u_1+u_2}^n}, \frac{z_{r_1+v_1+1}}{q_{r_1+v_1+1}^n}, \dots, \frac{z_{r_1+v_1+v_2}}{q_{r_1+v_1+v_2}^n}, \frac{Y'}{p_0^n}, \frac{Y'-Y_2}{3}, \frac{Y'-Z_2}{2} \right\}$$

$$B_3 = \left\{ \frac{z_1'}{p_1^n}, \frac{z_2'}{p_2^n}, \dots, \frac{z_{r_1}'}{p_{r_1}^n}, \frac{y_{r_1+u_1+u_2+1}}{p_{r_1+u_1+u_2+1}^n}, \dots, \frac{y_{r_2}}{p_{r_2}^n}, \frac{z_{r_1+v_1+v_2+1}}{q_{r_1+v_1+v_2+1}^n}, \dots, \frac{z_{r_3}}{q_{r_3}^n}, \frac{Z'}{p_{r_3}^n}, \frac{Z'-Y_3}{5}, \frac{Z'-Z_3}{3} \right\}$$

where $p_0, p_1, p_2, \dots, p_{r_2}\ q_{r_1+1}, \dots, q_{r_3}$ are different primes greater than 5.

Referring to the corresponding arguments in the proof of the Theorem 1 it is routine to show that the groups A_i and B_i $(i = 1, 2, 3)$ are indecomposable and that $A_1 + A_2 + A_3$ and $B_1 + B_2 + B_3$ are direct sums.

Now we come to find the conditions that should be imposed upon the matrix (a_{ik}) in order to be $A_1 + A_2 + A_3 = B_1 + B_2 + B_3$.

For every matrix (a_{ik}) the elements x'_j/p_j^n, y'_j/p_j^n, z'_j/p_j^n $(j=1,2,\dots,r_1)$ and X'/p_0^n, Y'/p_0^n, Z'/p_0^n of $B_1+B_2+B_3$ belong clearly to $A_1+A_2+A_3$ for any integer n. Since

$$\begin{aligned}
X'-Y_1 &= a_{11}X+(a_{12}+1)Y+a_{13}Z-(Y+Y_1),\\
X'-Z_1 &= a_{11}X+a_{12}Y+(a_{13}+1)Z-(Z+Z_1),\\
Y'-Y_2 &= a_{21}X+(a_{22}+1)Y+a_{23}Z-(Y+Y_2),\\
Y'-Z_2 &= a_{21}X+a_{22}Y+(a_{23}+1)Z-(Z+Z_2),\\
Z'-Y_3 &= a_{31}X+(a_{32}+1)Y+a_{33}Z-(Y+Y_3),\\
Z'-Z_3 &= a_{31}X+a_{32}Y+(a_{33}+1)Z-(Z+Z_3),
\end{aligned}$$

we infer that the elements $\dfrac{X'-Y_1}{2}$, $\dfrac{X'-Z_1}{5}$, $\dfrac{Y'-Y_2}{3}$, $\dfrac{Y'-Z_2}{2}$, $\dfrac{Z'-Y_3}{3}$, $\dfrac{Z'-Z_3}{3}$ belong to $A_1+A_2+A_3$ if only

$$\begin{aligned}
2 &\mid a_{11}, a_{12}+1, a_{13}, a_{21}, a_{22}, a_{23}+1,\\
3 &\mid a_{21}, a_{22}+1, a_{23}, a_{31}, a_{32}, a_{33}+1,\\
5 &\mid a_{31}, a_{32}+1, a_{33}, a_{11}, a_{12}, a_{13}+1.
\end{aligned}$$

These conditions are satisfied by the matrix of the form

$$(a_{ik})=\begin{pmatrix} 10\,i_1 & 5(2j_1-1) & 2(5k_1+2)\\ 6\,i_2 & 2(3j_2+1) & 3(2k_2-1)\\ 15\,i_3 & 3(5j_3-2) & 5(3k_3+1)\end{pmatrix},$$

where i_m, j_m, k_m $(m=1,2,3)$ are arbitrary integers. For any matrix of this form we have clearly $B_1+B_2+B_3\subseteq A_1+A_2+A_3$.

On the other hand if only $|\det(a_{ik})|=1$, the elements of $A_1+A_2+A_3$ of the form x_j/p_j^n, y_j/p_j^n, z_j/p_j^n, $j=1,2,\dots,r_1$ and X/p_0^n, Y/p_0^n, Z/p_0^n belong to $B_1+B_2+B_3$ for any integer n.

Let us denote $(b_{ik})=(a_{ik})^{-1}$. Since

$$\begin{aligned}
Y+Y_1 &= (b_{21}+1)X'+b_{22}Y'+b_{23}Z'-(X'-Y_1),\\
Y+Y_2 &= b_{21}+(b_{22}+1)Y'+b_{23}Z'-(Y'-Y_2),\\
Y+Y_3 &= b_{21}X'+b_{22}Y'+(b_{23}+1)Z'-(Z'-Y_3),\\
Z+Z_1 &= (b_{31}+1)X'+b_{32}Y'+b_{33}Z'-(X'-Z_1),\\
Z+Z_2 &= b_{31}X'+(b_{32}+1)Y'+b_{33}Z'-(Y'-Z_2),\\
Z+Z_3 &= b_{31}X'+b_{32}Y'+(b_{33}+1)Z'-(Z'-Z_3),
\end{aligned}$$

we see that the elements $\dfrac{Y+Y_1}{2}$, $\dfrac{Y+Y_2}{3}$, $\dfrac{Y+Y_3}{5}$, $\dfrac{Z+Z_1}{5}$, $\dfrac{Z+Z_2}{2}$, $\dfrac{Z+Z_3}{3}$ belong to $B_1+B_2+B_3$, if only

$$2 \mid b_{21}+1, b_{22}, b_{23}, b_{31}, b_{32}+1, b_{33},$$

$$3 \mid b_{21}, b_{22}+1, b_{23}, b_{31}, b_{32}, b_{33}+1,$$

$$5 \mid b_{21}, b_{22}, b_{23}+1, b_{31}+1, b_{32}, b_{33},$$

so that if

$$6 \mid b_{23}, b_{31}, \quad 10 \mid b_{22}, b_{33}, \quad 15 \mid b_{21}, b_{32},$$

$$b_{21} \equiv b_{32} \equiv 1 \pmod 2, \quad b_{22} \equiv b_{33} \equiv 2 \pmod 3, \quad b_{23} \equiv b_{31} \equiv 4 \pmod 5. \tag{3}$$

Suppose $\det(a_{ik}) = 1$. From (2) we infer that $i_1 \equiv 1 \pmod 3$, $i_2 \equiv 1 \pmod 5$ and $i_3 \equiv 1 \pmod 2$. For such $i_1\, i_2$ and i_3 the conditions (3) are satisfied as we can easily see.

Now there remains to be shown that the condition $|\det(a_{ik})| = 1$ can be realized. Supposing $i_1 = i_2, = i_3 = 1$, $j_1, = j_2, = j_3 = 0$ we obtain $\det(a_{ik}) = 31—30\,(22k_1 + 3k_2 — 25k_3)$, so that $\det(a_{ik}) = 1$ if, for example, $k_1 = 1$, $k_2 = —7$, $k_3 = 0$.

Thus in order to have $A_1 + A_2 + A_3 = B_1 + B_2 + B_3$ it is sufficient to use the matrix

$$(a_{ik}) = \begin{pmatrix} 10 & -5 & 14 \\ 6 & 2 & -45 \\ 15 & -6 & 5 \end{pmatrix}.$$

Note added in proof.

L. Fuchs informed me that A. L. S. Corner (Cambridge) proved the following theorem: for any integers $N, k\ (N > k)$ there is a torsionfree Abelian group G of rank N such that for $r_1 + \ldots + r_k = N$ there are the subgroups $A_1, \ldots, A_k$ of ranks $r_1, \ldots, r_k$ respectively, indecomposable and such that $G = \mathbf{A}_1 + \ldots + A_k$.

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MATHEMATICS

An Analogue of Fubini's Theorem and its Application to Random Linear Equations

by

C. RYLL-NARDZEWSKI

Presented by E. MARCZEWSKI on May 31, 1960

In this paper we shall prove a theorem concerning countable random sets. Roughly speaking, if a random set Z_ω, contained in a set Λ, is countable then after omitting a suitable countable subset Λ_0, we have for each point λ the probability of its belonging to the set Z_ω equal to zero. The precise formulation with all suppositions is given by Theorem 1.

We shall next consider the question of the necessity of our assumptions in Theorem 1 (see Example). In the last part (Theorem 2) of this paper an application is given to the spectral set of a wide class of random linear equations in Banach spaces.

1. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space with a σ-finite measure defined on σ-field $\mathcal{M}$ of subsets of Ω. Let $(\Lambda, \mathcal{B})$ be a *Borel space*, i.e. an arbitrary abstract set Λ for which there exists a 1—1 point transformation φ on a Borel set E (lying in a metric, separable and complete space) and the class $\mathcal{B}$ of subsets of Λ is mapped by φ on the class of all Borel subsets of E.

We emphasize that all uncountable Borel spaces are equivalent (by some point isomorphism), and in consequence we can always identify Λ with the unit-interval and $\mathcal{B}$ with the class of all its Borel subsets. It is easy to see that in Theorem 1 only the case of uncountable Λ is interesting and non-trivial.

Let Z be a measurable subset of product-space $\Omega \times \Lambda$. Measurability is understood with respect to the product field $\mathcal{M} \times_\sigma \mathcal{B}$ (= the smallest σ-field containing all "rectangles" $E \times F$, $E \in M$ and $F \in B$. We denote by Z_ω the sets: $\{\lambda : (\omega, \lambda) \in Z\}$ and $\{(\omega : (\omega, \lambda) \in Z\}$ respectively.

THEOREM 1. *If the sets Z_ω are countable for a.e. (in the sense of measure μ) points $\omega \in \Omega$, then we have $\mu(Z_\lambda) = 0$ for all $\lambda \in \Lambda$ with the exception of countably many λ's. In symbols*: card $\{\lambda : \mu(Z_\lambda) > 0\} \leqslant \aleph_0$.

Proof. We suppose, *a contrario*, that the set $Q = \{\lambda : \mu(Z_\lambda) > 0\}$ is uncountable. The function $\mu(Z_\lambda)$ of the variable λ is $\mathcal{B}$-measurable for every *rectangle* Z and consequently for any $Z \in \mathcal{M} \times_\sigma \mathcal{B}$. Hence Q is an uncountable $\mathcal{B}$-set. Then there exists some finite σ-measure ν defined on the field $\mathcal{B}$: (a) vanishing for all one point subsets of Λ and (b) positive for the set Q.

The existence of such a measure follows from the well-known facts:

1) Any uncountable Borel set contains a subset homeomorphic with the Cantor set (see e.g. [1] p. 355, Theorem 3),

2) For the Cantor set there is, of course, a measure with wanted properties.

Now, we apply Fubini's theorem to the product-measure $\mu \times \nu$:

$$(\mu \times \nu)(Z) = \int \mu(Z_\lambda)\, d\nu = \int_Q \mu(Z_\lambda)\, d\nu > 0,$$

and, on the other hand, in view of (a), we obtain

$$(\mu \times \nu)(Z) = \int \nu(Z_\omega)\, d\mu = 0.$$

This contradiction concludes the proof.

Remark: Our Theorem 1 can also be regarded as a certain generalisation of the well-known and simple fact that a random variable can admit only a countable set of value with positive probability. The same namely is valid for countably-multivalent random variables.

2. The purpose of this section is to give an example to verify that our suppositions imposed on the set Λ and the σ-field $\mathcal{B}$ are really indispensable.

Example. We put the real line with Lebesgue measure as a measure space $(\Omega, \mathcal{M}, \mu)$ and denote by Λ the famous Lusin set which is uncountable and whose intersection with each everywhere non-dense set is at most countable (see e.g. [1] p. 432). We denote by $\mathcal{B}$ the class of all relatively Borel subsets of the set Λ.

We define the set Z as follows:

$$Z \stackrel{\text{df}}{=} \{(\omega, \lambda : \omega + \lambda \in E, \omega \in \Omega, \lambda \in \Lambda\},$$

where E is a linear nowhere dense set of positive Lebesgue measure. It is easy to see that all conditions imposed on the set Z by Theorem 1 are fulfilled. On the other hand we have $\mu(Z_\lambda) = |E| > 0$ for all λ belonging to Λ. Hence, the condition for $(\Lambda, \mathcal{B})$ to be a Borel space is in fact essential.

3. In this section we give a certain application of Theorem 1 to the theory of linear equations in Banach spaces. We consider an operator-function $\{T_\omega\}$ defined on the measure space $(\Omega, \mathcal{M}, \mu)$ (having the same sense as in the 1-st section). For each fixed point $\omega \in \Omega$ $\{T_\omega\}$ is a linear operator from a separable, complex Banach space X into itself.

We have

THEOREM 2. *If*

(1) $\{T_\omega\}$ *is weakly measurable, i.e. the function* $x_0^* T_\omega x_0$ *of variable* ω *is measurable for all* $x_0^* \in X^*$ and $x_0 \in X$, and

(2) *for a.e.* $\omega \in \Omega$ *the set* S_ω *of the complex numbers for which a bounded inverse of operator* $(I - \lambda T_\omega)$ *does not exist, is countable (this set* S_ω *is called a spectral set of operator* T_ω*),*

then the spectral set S *of operator-function* $\{T_\omega\}$ *(i.e. the set of* λ*'s for which a bounded inverse* $(I - \lambda T_\omega)^{-1}$ *on any set of* ω*'s of positive measure does not exist, is also countable.*

Proof. Now, we assume that the set Λ (from Theorem 1) is identical with the complex plane, and the class $\mathcal{B}$ is the σ-field of all Borel sets of complex numbers.

It suffices evidently to justify the measurability of the set:

$$Z \overset{\mathrm{df}}{=} \{(\omega, \lambda) \text{ a bounded inverse of } (I - \lambda T_\omega) \text{ does not exist}\}.$$

Let $\{x_n\}$ be any sequence dense in the space X. It is not difficult to verify the following formula:

$$Z = \bigcup_{p=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{(\omega, \lambda) : \|x_n - \lambda T_\omega x_n\| \geq \frac{1}{p} \|x_n\|\right\} \cap$$
$$\cap \bigcap_{n=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcup_{m=1}^{\infty} \left\{(\omega, \lambda) : \|x_n - (x_m - \lambda T_\omega x_m)\| < \frac{1}{p}\right\}.$$

It must be remembered that the weak measurability of $\{T_\omega\}$ implies a strong one (cf. [2], p. 74) in separable Banach space. Hence, the set Z is measurable, i.e. it belongs to the field $\mathcal{M} \times_\sigma \mathcal{B}$.

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MATHEMATICS

On the Strong Summability of Orthogonal Series

by

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Presented by W. ORLICZ on May 31, 1960

A series

(1) $$A_1 + A_2 + \cdots + A_n + \cdots$$

with partial sums s_n is said to be strongly summable (R, 1) ((R, 1) indicates here Riesz's logarithmic mean of order 1) to the sum s, if

$$\sum_{k=1}^{n} \frac{1}{k} |s_k - s| = o(\log n), \quad \text{as} \quad n \to \infty.$$

It is strongly summable (R, 1) with index p $(p = 1, 2, \ldots)$ to the sum s, if

$$\sum_{k=1}^{n} \frac{1}{k} |s_k - s|^p = o(\log n), \quad \text{as} \quad n \to \infty.$$

Further, we shall consider the strong summability (R, 1) of orthogonal series by $p = 2$.

Suppose that $ON\{\varphi_n(x)\}$ denotes an orthonormal system, defined in $\langle 0, 1 \rangle$ and that

(2) $$\sum_{n=1}^{\infty} c_n \varphi_n(x)$$

denotes the orthogonal series, in which the coefficients $\{c_n\} \in l^2$, i. e.

(3) $$\sum_{n=1}^{\infty} c_n^2 < \infty.$$

We shall denote by $s_n(x)$ the partial sums of series (2).

The results obtained here are analogous to those published recently by K. Tandori for the method (C, 1) ([2], Mitt. II, IV, VI).

THEOREM 1. *Let $\{c_n^*\} \in l^2$ be a sequence of real, positive numbers, satisfying the inequality*

$$\sqrt{n \log n}\, c_n^* \geqslant \sqrt{(n+1) \log (n+\iota)}\, c_{n+1}^* \, (n = 2, 3, \ldots)$$

and $\{c_n\}$ an arbitrary sequence of real numbers such, that

$$c_n = O(c_n^*).$$

If the series (2) *is by these coefficients almost everywhere in* $\langle 0,1\rangle$ *summable* (R, 1) *to a certain function* $f(x)$, *then*

$$\sum_{k=1}^{n} \frac{1}{k} [s_{v_k}(x) - f(x)]^2 = o(\log n), \quad \text{as } n \to \infty,$$

for every increasing sequence of indices $\{v_k\}$, *almost everywhere in* $\langle 0,1\rangle$

Denoting

$$\tau_n^{(v)} = \frac{1}{\log n} [s_{v_1}(x)/1 + s_{v_2}(x)/2 + \cdots s_{v_n}(x)/n],$$

we formulate the following lemmas:

LEMMA 1. *The sequence* $\{\tau_n^{(v)}(x)\}$ *is almost everywhere convergent, if and only if the sequence* $\{s_{v_{2^{2^n}}}(x)\}$ *is also almost everywhere convergent.*

LEMMA 2. *The sequence* $\{\tau_n^{(v)}(x)$ *is almost everywhere convergent to a certain function* $f(x)$ *if and only if*

$$\sum_{k=1}^{n} \frac{1}{k} [s_{v_k}(x) - f(x)] = o(\log n), \quad \text{as } n \to \infty,$$

for every sequence of increasing indices $\{v_k\}$, *almost everywhere in* $\langle 0,1\rangle$.

THEOREM 2. *There exist a system* $ON\{\Phi_n(x)\}$ *in* $\langle 0,1\rangle$, *a sequence of coefficients* $\{c_n\} \in l^2$ *and an increasing sequence of indices* $\{v_k\}$, *such that the series*

$$\sum_{n=1}^{\infty} c_n \Phi_n(x) \tag{4}$$

is summable (R, 1) *almost everywhere in* $\langle 0,1\rangle$ *to certain function* $f(x)$. *but the sequence*

$$\frac{1}{\log n} [S_{v_1}(x)/1 + S_{v_2}(x)/2 + \cdots + S_{v_n}(x)/n]$$

is in $\langle 0,1\rangle$ *almost everywhere divergent,* $S_k(x)$ *denoting here the k-partial sums of* (4).

Remark: Basing on the evident inequality

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} S_{v_k}(x) - f(x)$$

$$\sqrt{\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} [S_{v_k}(x) - f(x)]^2 \eta_n} + |f(x)| \cdot |\eta_n - 1|,$$

where

$$\eta_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \to 1, \quad \text{as } n \to \infty,$$

and Theorem 2, we may also say that in this case

$$\sum_{k=1}^{n} \frac{1}{k} [S_{v_k}(x) - f(x)]^2 \neq o(\log n)$$

almost everywhere in $\langle 0, 1 \rangle$.

If, however, $v_k = k$ ($k = 1, 2, \ldots$) and series (4) is almost everywhere summable $(R, 1)$ to a certain function $f(x) \in L^2$, then

$$\sum_{k=1}^{n} \frac{1}{k} [S_{v_k}(x) - f(x)]^2 = o(\log n), \quad \text{as } n \to \infty,$$

almost everywhere ([1], see proof of Th. 1 p. 15).

THEOREM 3. *If*

$$\sum_{n=8}^{\infty} c_n^2 (\log \log \log n)^2 < \infty,$$

then there exists such a function $f(x) \in L^2$ that

$$\sum_{k=1}^{n} \frac{1}{k} [s_{v_k}(x) - f(x)]^2 = o(\log n), \quad \textit{as } n \to \infty,$$

for every sequence of indices $\{v_k\}$ in $\langle 0, 1 \rangle$ almost everywhere.

The proofs of the above given theorems will be published in Studia Mathematica.

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MATHEMATICS

A Note on Sentences Preserved Under Direct Products and Powers

by

M. MACHOVER

Presented by A. MOSTOWSKI on June 1, 1960

In this note *) we prove theorems concerning the recursive-enumerability and recursivity of the classes of sentences which are preserved under direct products or powers of arbitrary or pre-assigned "lengths"

Theorem 1 has been proved in [7] for some of the classes considered here, with the notable exception of the classes denoted below by B_∞ and A_∞.

Theorem 2 was stated (without proof and without specification of the language concerned) in [2], again with the exception of B_∞ and A_∞.

Let L be a simple applied functional calculus with equality (in the sense of [1]). All relational systems considered are assumed to be appropriate to L. A sentence (i.e., a wff without free variables) in L is either true or false in a given relational system. In the former case we say that the given relational system is a model of the sentence.

Consider the following classes of sentences in L:

$A_k =$ the class of sentences preserved under direct products of k systems (i.e. sentences which, if true in each of k relational systems, are true in their direct product), $k = 2, 3, \ldots$,

$A_{\text{fin}} = \bigcap_{k=2}^{\infty} A_k$,

$A_\infty =$ the class of sentences preserved under infinite direct products,

$A = A_\infty \cap A_{\text{fin}}$,

$B_k =$ the class of sentences preserved in the k—th power of systems, $k = 2, 3, \ldots$,

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*) This note is a part of a survey of results concerning sentences preserved under various model-theoretic operations, prepared while the author was holding a fellowship of Warsaw University, granted by the Polish Government under the sponsorship of UNESCO. The author wishes to thank Prof. A. Mostowski for valuable comments and for bringing [2], [5], [7] and [8] to his attention.

$B_{\text{fin}} = \bigcap_{k=2}^{\infty} B_k,$

$B_\infty =$ the class of sentences preserved in infinite powers of systems,

$B = B_{\text{fin}} \cap B_\infty.$

We shall make use of the following:

LEMMA 1. *For each sentence Φ of L, one can effectively determine positive integers m and n, sentences θ_α $(\alpha = 1, \ldots, m)$ in L, indices $i_{\alpha,\beta}$ (each either $= 0$ or $= 1$) and non-negative integers $h_{\alpha,\beta}$ $(\alpha = 1, \ldots, m$ $\beta = 1, \ldots, n)$ such that in any relational system exactly one of the θ_α is true and such that given a family F of relational systems, Φ will be true in their product if and ony if*

(1) $$\prod_{\beta=1}^{m} \sum_{\alpha=1}^{m} \{\theta_\alpha; i_{\alpha,\beta}, h_{\alpha,\beta}\}_F = 0$$

where $\{\theta; 0, h\}_F$; $(\{\theta; 1, h\}_F)$ is $= 0$ if θ is true in at least (exactly) h systems of F, and $= 1$ otherwise.

This was proved in [6] for the case where all systems in F are identical (i.e. for direct powers) but may be shown to hold in the general case too, (cf. [3]).

We prove the following consequence of Lemma 1:

LEMMA 2. *For any sentence Ψ of L one can effectively determine an integer k such that, for any $l \geqslant k$, $\Psi \in A_l$ if and only if $\Psi \in A_\infty$.*

Proof. Let Φ be the sentence $\sim \Psi$. Determine $m, n, \theta_\alpha, i_{\alpha,\beta}, h_{\alpha,\beta}$ of Lemma 1. Take $k = (\sum_{\alpha,\beta} h_{\alpha,\beta}) + 2$. Suppose $\Psi \in A_l$, where $l \geqslant k$. Let F be an infinite family of models of Ψ. We have to show that Ψ is true in the product of the members of F. Assume Ψ is false in the product. Therefore Φ is true in it. By Lemma 1 it follows that (1) holds. In other words, for some β_0 we must have

(2) $$\{\theta_\alpha; i_{\alpha,\beta_0}, h_{\alpha,\beta_0}\}_F = 0 \quad \text{for} \quad \alpha = 1, \ldots, m.$$

But each system is a model of exactly one of the θ_α. Consequently, one of θ_α say θ_1, is true in infinitely many members of F. Then i_{1,β_0} must be 0. Because of (2) and the fact that $l > \sum_{\alpha=1}^{m} h_{\alpha,\beta_0}$, there is a subfamily F_0 of F having l members, $l - \sum_{\alpha=2}^{m} h_{\alpha,\beta_0}$ of which are models of θ_1, and h_{α,β_0} of which are models of $\theta_\alpha (\alpha = 2, \ldots, m)$.

Since $i_{1,\beta_0} = 0$ and $h_{1,\beta_0} < l - \sum_{\alpha=2} h_{\alpha,\beta_0}$, it follows that

$$\{\theta_\alpha; i_{\alpha,\beta_0}, h_{\alpha,\beta_0}\}_{F_0} = \{\theta_\alpha; i_{\alpha,\beta_0}, h_{\alpha,\beta_0}\}_F \ (\alpha = 1, \ldots, m)$$

and Φ must be true in the product of these l systems, contradicting our assumption that $\Psi \in A_l$.

Now let $\Psi \epsilon A_{\infty}$ and let F be a family of l models of Ψ. If their product is not a model of Ψ, we have (2) for some β. Since $l > \sum_{\alpha=1}^{m} h_{\alpha,\beta_0}$, one of the θ_α must be true in more than h_{α,β_0} systems of F and we may assume this to be the case for θ_1. Again it follows that $i_{1,\beta_0} = 0$. Add to F infinitely many systems which are models of both Ψ and θ_1, getting an infinite family F_0. The rest of the argument is as in the first part of the proof.

LEMMA 3. *For each sentence Ψ there can be effectively found an integer k such that, for any $l \geqslant k$, $\Psi \epsilon B_l$ is equivalent to $\Psi \epsilon B_\infty$.*

The proof of this is simpler than that of the previous lemma.

As immediate results of the last two lemmata we have

$$A_{\mathrm{fin}} = A, \qquad B_{\mathrm{fin}} = B.$$

We also note that $A_2 = A_{\mathrm{fin}}$ easily follows from the definition of these classes.

THEOREM 1. *The classes $A_k, B_k, B_{\mathrm{fin}}, A_\infty, B_\infty$ are recursively enumerable and if L is singular (i.e. has only one place predicate letters) then they are recursive.*

Proof. We extend L (by augmenting its list of primitive symbols) adding a list of new one-place predicate letters $M^1, M^2, \ldots$ and, for each variable x, a list of new variables $x^1, x^2, \ldots$. Given a sentence Φ in L and an integer $k \geqslant 2$, we replace in Φ each part of the form $(x = y)$ by $(x^1 = y^1)$ & ... & $(x^k = y^k)$, each part of the form $R(x, y, \ldots)$ by $R(x^1, y^1, \ldots)$, & ... & $R(x^k, y^k, \ldots)$ each (Ex) by $(Ex^1) \ldots (Ex^k)$ and each (x) by $(x^1) \ldots (x^k)$. Relativize each (Ex^h) and (x^h) to $M^h, h = 1, 2, \ldots, k$. Call the resulting sentence Φ_k^* and let $\Phi^{(h)}$ be the sentence resulting from Φ by relativizing quantifiers to M^h. Let Φ_k be the sentence

$$(Ex)\,M^1(x)\,\&\ldots\&\,(Ex)\,M^k(x)\,\&\,\Phi^{(1)}\,\&\ldots\&\,\Phi^{(k)} \supset \Phi_k^*.$$

Clearly, Φ_k is provable if and only if $\Phi \epsilon A_k$.

Similarly, we construct a sentence Φ^k, which is provable if and only if $\Phi \epsilon B_k$.

This proves the Theorem for A_k, B_k since the extended language has only one-place predicate letters beside the predicate letters of L, and thus the decision problem for it is solvable if (and only if) L is singular.

For B_{fin} the Theorem follows from Lemma 3, since $\Phi \epsilon B_{\mathrm{fin}} \equiv \Phi \epsilon B_2$ and ... and $\Phi \epsilon B_k$, where k can be found effectively from Φ. With the same k we also have $\Phi \epsilon B_k \equiv \Phi \epsilon B_\infty$, which proves the theorem for B_∞. Similarly for A_∞, using Lemma 2.

THEOREM 2. *If among the primitive symbols of L there are*

(a) *two predicate letters, one of which has at least two argument--places*

or

(b) *a predicate letter of more than two argument-places,*

then $A_k, B_k, B_{\mathrm{fin}}, A_\infty, B_\infty$ *are all non-recursive.*

Proof. First let us consider case (a).

Suppose L has predicate letters P, R of m, n argument places respectively and $m \geqslant 2$. Let C be the class of all sentences of L which do not contain equality, nor any predicate letter different from P. Take any $\Phi \epsilon C$ and let Ψ be the sentence

$$\Phi \,\&\, (Ex)(Ey)[(x \neq y) \,\&\, (z)\{R(z, \ldots, z) \equiv (z = x) \vee (z = y)\}].$$

If Φ is not satisfiable, then neither is Ψ, so that Ψ belongs to all the classes $A_k, B_k, B_{\mathrm{fin}}\, A_\infty$ and B_∞. If Φ is satisfiable, then obviously Ψ cannot belong to any of these classes. As the decision problem for formulae of C is known to be unsolvable, the theorem is proved in case (a).

Now take case (b). It will not limit generality to assume that L has a three-place predicate letter Q. Let P be a two-place predicate letter, not necessarily a symbol of L, and again let C be the class of all first--order sentences containing neither equality, nor any predicate letter different from P. Let $\Phi \epsilon C$. In Φ replace each part of the form $P(x, y)$ by $(z)\,[(z \neq y) \supset Q(x, y, z)]$ and let Φ^* be the resulting sentence. Let Ψ be the sentence

$$(z)(w)(x)(y)[(z \neq y) \,\&\, (w \neq y) \supset \{Q(x, y, z) \equiv Q(x, y, w)\} \,\&,$$

$$\Phi^* \,\&\, (Ex)(Ey)[(x \neq y) \,\&\, (z)\{Q(z, z, z) \equiv (z = x) \vee (z = y)\}].$$

Now continue as in the first part of the proof.

Remark 1. The case where L has just one predicate letter, which has two argument places remains open.

Remark 2. The method of proof of Theorem 2 is applicable also to a language without equality, provided (a) and (b) are strengthened a little.

Remark 3. In [8] it is proved that the classes of sentences preserved in subsystems and in unions of chains respectively are not recursive. Using the method of the proof of Theorem 2, it is possible to prove these facts in a simpler way (in particular without using results of [4], [9] and the solvability of the decision problem for universal--existential sentences).

However, more stringent conditions than are necessary for the proofs in [8] have to be laid down concerning L: it has to have equality, and, in the case of chains, to have also at least two two-place predicate letters or at least one predicate letter of more than two places. An analogous theorem concerning sentences preserved in homomorphic images may be found in [5].

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MATHÉMATIQUE

Propriétés d'une intégrale singulière généralisée de Cauchy

par

W. ŻAKOWSKI

Présenté par T. WAŻEWSKI le 8 juin 1960

1. Introduction et formules principales. Soit dans le plan de la variable complexe un ensemble $P = \sum_{\sigma} \widehat{c_\sigma c_{\sigma'}}$, de points, composé d'un nombre fini d'arcs simples $\widehat{c_\sigma c_{\sigma'}}$ et un ensemble $C = \sum_{r} C_r$ de lignes fermées de Jordan. Les extrémités des arcs $\widehat{c_\sigma c_{\sigma'}}$, peuvent appartenir à un seul arc, ou bien elles peuvent être communes à plusieurs arcs différents. En tout cas les arcs $\widehat{c_\sigma c_{\sigma'}}$, et les lignes C_r n'ont pas de points intérieurs communs. Désignons par $c_1, c_2, \ldots, c_p$ les extrémités des arcs donnés. Nous supposons que les arcs $\widehat{c_\sigma c_{\sigma'}}$, et les contours C_r ont des tangentes continues en tout point, même aux extrémités. Les points $c_1, c_2, \ldots, c_p$ sont donc soit des points anguleux, soit des points de rebroussement, soit des points multiples, soit des simples extrémités des lignes formées par les arcs donnés. Nous admettons que tout arc et contour a une direction individuelle, fixée indépendamment des autres lignes. On admet que les deux côtés, positif et négatif, de toute coupure sont en telle relation avec la direction positive de la coupure que les demi-plans $\mathrm{Im}(z) > 0$ et $\mathrm{Im}(z) < 0$ avec la direction positive de l'axe réel. Nous présenterons maintenant les définitions des classes de fonctions complexes discontinues, définies sur l'ensemble de points $L = C + P$, données par W. Pogorzelski [1].

Définition. On appelle classe $\mathfrak{H}_\alpha^\mu$ un ensemble de toutes les fonctions complexes $\varphi(t)$, définies en tout point t de l'ensemble L, différent des points de discontinuité $c_1, c_2, \ldots, c_p$, qui vérifient les inégalités suivantes

$$(1) \qquad |\varphi(t)| < \frac{\text{const}}{\prod_{\alpha=1}^{p} |t - c_\sigma|^\alpha}, \qquad \text{et} \qquad |\varphi(t) - \varphi(t_1)| < \frac{\text{const}\, |t - t_1|^\mu}{|t - c_\sigma|^{\alpha+\mu} |t_1 - c_{\sigma'}|^{\alpha+\mu}},$$

à l'intérieur de tout arc $\widehat{c_\sigma c_{\sigma'}}$, composant de P, le point t_1 étant situé sur l'arc $\widehat{t c_{\sigma'}}$. Si $t, t_1 \in C_r$, nous supposons que $c_\sigma = c_{\sigma'} = c^*$, où c^* est un point de discontinuité fixé arbitrairement. On admet que les paramètres réels α et μ, fixés pour la classe donnée, vérifient les inégalités

$$0 \leqslant \alpha < 1, \quad 0 < \mu < 1, \quad \alpha + \mu < 1.$$

Les propriétés des classes $\mathfrak{H}_\alpha^\mu$ étaient étudiées dans les travaux [1]—[3].

Supposons que le point t est situé à intérieur d'une ligne dirigée $l = \widehat{ab}$ ($a = b$, si $t \epsilon C_r$), et $\varphi(\tau) \epsilon \mathfrak{H}_\alpha^\mu$. Nous considérons maintenant deux points intérieurs t_1 et t_2 de l'arc $\widehat{ab}$ tels, que le point t soit situé à l'intérieur de l'arc $\widehat{t_1 t_2}$, ($t_2 \epsilon \widehat{tb}$).

Si la limite suivante existe

$$\lim_{t_1, t_2 \to t} \int\limits_{L - \widehat{t_1 t_2}} \frac{\varphi(\tau)\, d\tau}{\tau - t}, \quad \text{où} \quad \lim_{t_1, t_2 \to t} \left| \frac{t_2 - t}{t_1 - t} \right| = 1, \tag{2}$$

nous la désignerons par $\int\limits_L \frac{\varphi(t)}{\tau - t} d\tau$ et nous l'appellerons *intégrale singulière de Cauchy.*

2. Intégrale singulière généralisée de Cauchy. Supposons que la fonction $k(t)$ est définie pour $t \epsilon L - \sum\limits_\sigma c_\sigma$ et admet les valeurs positives.

Définition. Si la limite suivante existe

$$\lim_{t_1, t_2 \to t} \int\limits_{L - \widehat{t_1 t_2}} \frac{\varphi(\tau)\, d\tau}{\tau - t}, \quad \text{où} \quad \lim_{t_1, t_2 \to t} \left| \frac{t_2 - t}{t_1 - t} \right| = k(t), \tag{3}$$

nous la désignerons par $\int\limits_L^{(k)} \frac{\varphi(\tau)}{\tau - t} d\tau$ et l'appellerons *intégrale singulière généralisée de Cauchy.*

Écrivons la décomposition suivante

$$\int\limits_L^{(k)} \frac{\varphi(\tau)\, d\tau}{\tau - t} = \int\limits_{L-l} \frac{\varphi(\tau)\, d\tau}{\tau - t} + \int\limits_l \frac{\varphi(\tau) - \varphi(t)}{\tau - t} d\tau + \varphi(t) \int\limits_l^{(k)} \frac{d\tau}{\tau - t}. \tag{4}$$

Nous aurons, d'après l'égalité

$$\int\limits_l^{(k)} \frac{d\tau}{\tau - t} = \lim_{t_1, t_2 \to t} \left(\log \frac{b - t}{a - t} - \log \left| \frac{t_2 - t}{t_1 - t} \right| - \arg \frac{t_2 - t}{t_1 - t} \right) = \int\limits_l \frac{d\tau}{\tau - t} - \log k(t) \tag{5}$$

l'égalité suivante

$$\int\limits_L^{(k)} \frac{\varphi(\tau)\, d\tau}{\tau - t} = \int\limits_L \frac{\varphi(\tau)\, d\tau}{\tau - t} - \varphi(t) \log k(t). \tag{6}$$

3. Théorème généralisé de Pogorzelski. Une propriété principale des fonctions de classe $\mathfrak{H}_\alpha^\mu$, exprime le théorème suivant, démontré par W. Pogorzelski: la transformation singulière de Cauchy $\int\limits_L \frac{\varphi(\tau)\, d\tau}{\tau - t} = \Psi(t)$, transforme toute fonction $\varphi(t)$ de classe $\mathfrak{H}_\varepsilon^\mu$ en fonction $\Psi(t)$ de la même classe si $\alpha > 0$, et en fonction de classe $\mathfrak{H}_\alpha^\mu$, si $\alpha = 0$, (ε désigne une constante positive arbitrairement petite (voir [1] et [2]).

THÉORÈME 1. *Si* $\varphi(t)\in\mathfrak{H}^{\mu_1}_{\alpha_1}$, $k(t)\in\mathfrak{H}^{\mu_2}_{\alpha_2}$, $\alpha_1+\alpha_2+\min(\mu_1,\mu_2)<1$ et $\inf\limits_{L} k(t)=k_0>0$, *alors la transformation singulière de Cauchy généralisée*

$$\Psi(t)=\int\limits_L^{(k)}\frac{\varphi(\tau)}{\tau-t}\,d\tau,$$

appartient à la classe $\mathfrak{H}^{\mu}_{\varrho}$, *où* $\mu=\min(\mu_1,\mu_2)$, $\varrho=\alpha_1+\alpha_2$ *si* $\alpha_2>0$ *et* $\varrho=\begin{smallmatrix}\alpha_1,\ \mathrm{si}\ \alpha_1>0\\ \varepsilon,\ \mathrm{si}\ \alpha_1=0\end{smallmatrix}$ *(ε désigne une constante positive arbitrairement petite), si* $\alpha_2=0$.

Démonstration. Elle résulte facilement de l'égalité (6) et du théorème de Pogorzelski cité.

4. **Transformation de Poincaré-Bertrand généralisée.** On appelle transformation de Poincaré-Bertrand généralisée l'égalité suivante, (voir [4]),

$$\int\limits_L\frac{d\tau}{\tau-t}\int\limits_L\frac{f(t,\tau,s)\,\varphi(s)}{s-\tau}\,ds=-\pi^2 f(t,t,t)\,\varphi(t)+\int\limits_L ds\int\limits_L\frac{f(t,\tau,s)\,\varphi(s)}{(\tau-t)(s-\tau)}\,d\tau, \tag{7}$$

où la fonction complexe $f(t,\tau,s)$ est définie dans la région $t\in L$, $\tau\in L$, $s\in$ (les extrémités des arcs $\widehat{c_\sigma c_{\sigma'}}$, exceptés) et vérifie la condition de Hölder relativement à t, τ et s, (séparément sur les lignes d'ensemble L), et en outre $\varphi(t)\in\mathfrak{H}^{\mu}_{\alpha}$. Les intégrales singulières dans (7), ont le sens singulier de Cauchy. Nous écrivons

$$I_1=\int\limits_L\frac{d\tau}{\tau-t}\int\limits_L\frac{f(t,\tau,s)\,\varphi(s)}{s-\tau}\,ds,\ \text{et}\ I_2=\int\limits_L ds\int\limits_L\frac{f(t,\tau,s)\,\varphi(s)}{(\tau-t)(s-\tau)}\,d\tau. \tag{8}$$

Par une méthode analogue à la démonstration de l'égalité (6), nous aurons les égalités

$$\begin{aligned}I_1&=\int\limits_L\frac{d\tau}{\tau-t}\left[\int\limits_L^{(k)}\frac{f(t,\tau,s)}{s-\tau}\varphi(s)\,ds+f(t,\tau,\tau)\,\varphi(\tau)\log k(\tau)\right]=\\&=\int\limits_L^{(k)}\frac{d\tau}{\tau-t}\left[\int\limits_L^{(k)}\frac{f(t,\tau,s)}{s-\tau}\varphi(s)\,ds+f(t,\tau,\tau)\,\varphi(\tau)\log k(\tau)\right]+\\&\quad+\log k(t)\left[\int\limits_L^{(k)}\frac{f(t,t,s)}{s-t}\varphi(s)\,ds+f(t,t,t,)\,\varphi(t)\log k(t)\right]=\\&=\int\limits_L^{(k)}\frac{ds}{\tau-t}\int\limits_L^{(k)}\frac{f(t,\tau,s)}{s-\tau}\varphi(s)\,ds+\int\limits_L^{(k)}\frac{f(t,\tau,\tau)\,\varphi(\tau)\log k(\tau)}{\tau-t}\,d\tau+\\&\quad+\log k(t)\int\limits_L^{(k)}\frac{f(t,t,s)}{s-t}\varphi(s)\,ds+f(t,t,t)\,\varphi(t)\log^2 k(t),\end{aligned} \tag{9}$$

et

$$(10)\qquad I_2=\int_L\Big[\int_L\frac{f(t,\tau,s)}{\tau-t}d\tau-\int_L\frac{f(t,\tau,s)}{\tau-s}d\tau\Big]\frac{\varphi(s)}{s-t}ds=$$

$$=\int_L\Big[\int_L^{(k)}\frac{f(t,\tau,s)}{\tau-t}d\tau-\int_L^{(k)}\frac{f(t,\tau,s)}{\tau-s}d\tau\Big]\frac{\varphi(s)}{s-t}ds+\log k(t)\int_L\frac{f(t,t,s)\varphi(s)}{s-t}ds-$$

$$-\int_L\frac{f(t,s,s)\varphi(s)\log k(s)}{s-t}ds=\int_L^{(k)}ds\int_L^{(k)}\frac{f(t,\tau,s)\varphi(s)}{(\tau-t)(s-\tau)}d\tau+$$

$$+\log k(t)\int_L^{(k)}\frac{f(t,t,s)\varphi(s)}{s-t}ds-\int_L^{(k)}\frac{f(t,s,s)\varphi(s)\log k(s)}{s-t}ds,$$

pour la fonction $k(t)\in\mathfrak{H}_\alpha^\mu,(2\alpha+\mu<1,\ \inf_L k(t)=k_0>0)$, arbitrairement fixée.

Il en résulte, d'après les égalités (7), (9) et (10) que

$$(11)\qquad \int_L^{(k)}\frac{d\tau}{\tau-t}\int_L^{(k)}\frac{f(t,\tau,s)\varphi(s)}{s-\tau}ds=-[\pi^2+\log^2k(t)]f(t,t,t)\varphi(t)-$$

$$-2\int_L^{(k)}\frac{f(t,s,s)\varphi(s)\log k(s)}{s-t}ds+\int_L^{(k)}ds\int_L^{(k)}\frac{f(t,\tau,s)\varphi(s)}{(\tau-t)(s-\tau)}d\tau.$$

L'égalité (11) est la transformation de Poincaré-Bertrand généralisée au sens (3), pour les intégrales singulières généralisées de Cauchy. Dans le cas particulier $k(t)\equiv 1$, nous aurons l'égalité (7).

5. **Les formules de Plemelj et leur forme invariante.** Supposons que les lignes d'ensemble L admettent un nombre fini des points distingués (points anguleux, soit points de rebroussement). Si $\varphi(t)\in\mathfrak{H}_\alpha^\mu$, alors la fonction définie par l'intégrale du type Cauchy

$$(12)\qquad \Phi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)d\tau}{\tau-z},$$

est une fonction séparément holomorphe en dehors des coupures L. D'après les formules connues de Plemelj, les valeurs limites $\Phi^+(t)$, $\Phi^-(t)$, (des deux côtés des coupures, voir [1]), existent en tout point $t\in L$, (les extrémités des arcs exceptés), et sont les suivantes:

$$(13)\qquad \begin{aligned}\Phi^+(t)&=\Big(1-\frac{a_t}{2\pi}\Big)\varphi(t)+\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)d\tau}{\tau-t},\\ \Phi^-(t)&=-\frac{a_t}{2\pi}\varphi(t)=-\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)d\tau}{\tau-t},\end{aligned}$$

où les intégrales ont le sens singulier de Cauchy; a_t désigne la limite de l'angle formé par le vecteur $t_1 - t$ avec le vecteur $t_2 - t$, si $t_1, t_2 \to t$,

(14) $$a_t = \lim_{t_1, t_2 \to t} \arg \frac{t_1 - t}{t_2 - t},$$

(voir (5)), $0 \leqslant a_t \leqslant 2$. D'après l'égalité (6) nous aurons les formules suivantes

(15) $$\Phi^+(t) = \left[1 - \frac{a_t}{2\pi} + \frac{\log k(t)}{2\pi i}\right]\varphi(t) + \frac{1}{2\pi i}\int_L^{(k)} \frac{\varphi(\tau)\,d\tau}{\tau - t},$$
$$\Phi^-(t) = \left[-\frac{a_t}{2\pi} + \frac{\log k(t)}{2\pi i}\right]\varphi(t) + \frac{1}{2\pi i}\int_L^{(k)} \frac{\varphi(\tau)\,d\tau}{\tau - t}.$$

Les formules (15), dans le cas particulier où $k(t) = k = \text{const} > 0$ et $\varphi(t)$ vérifie la condition de Hölder, étaient étudiées par F. Gachov [5].

D'après (4) et (5), nous écrivons les formules (15) sous la forme invariante

(16) $$\Phi^+(t) = \frac{1}{2\pi i}\int_{L-l} \frac{\varphi(\tau)\,d\tau}{\tau - t} + \frac{1}{2\pi i}\int_l \frac{\varphi(\tau) - \varphi(t)}{\tau - t}\,d\tau + \frac{\varphi(t)}{2\pi i}\log\frac{b - t}{a - t} + \varphi(t),$$
$$\Phi^-(t) = \frac{1}{2\pi i}\int_{L-l} \frac{\varphi(\tau)\,d\tau}{\tau - t} + \frac{1}{2\pi i}\int_l \frac{\varphi(\tau) - \varphi(t)}{\tau - t}\,d\tau + \frac{\varphi(t)}{2\pi i}\log\frac{b - t}{a - t},$$

(où le point t est situé à l'intérieur d'une ligne dirigée l — voir introduction), c'est-à-dire, que les formules (16) ne dépendent pas de la fonction $k(t)$. Nous remarquons que les intégrales du membre droit (16) existent dans le sens ordinaire, comme les intégrales à faible singularité.

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MATHEMATICS

A Generalization of Certain Extension Theorems

by

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Presented by W. ORLICZ on June 13, 1960

1. In this note we prove a generalization 3 of a theorem of Mazur and Orlicz ([1], theorem 2.41) to the case of convex functionals $\omega(x)$ which may assume infinite values, instead of Banach functionals. The Hahn-Banach extension theorem for convex functionals follows immediately as an application (see Theorem 4). The latter was formulated by H. Nakano [2] and plays a role in the theory of modular spaces.

Let X be a real linear space and let $\omega(x)$ be a functional defined on X and satisfying the following conditions:

i. $$-\infty < \omega(x) \leqslant \infty \quad \text{for every} \quad x \in X,$$

ii. $$\omega(\alpha x + \beta y) \leqslant \alpha\omega(x) + \beta\omega(y) \ \text{for} \ \alpha, \beta > 0, \ \alpha + \beta = 1,$$

iii. $$\text{if for an } x \in X, \ \omega(tx) = \infty \ \text{for all} \ t > 0,$$

$$\text{then} \ \omega(tx) = \infty \ \text{for all} \ t \neq 0.$$

Assuming $\omega(0) < \infty$, let us write

$$X_* = \{x \in X : \omega(tx) < \infty \ \text{for some} \ t \neq 0\}.$$

Clearly, *X_* is a linear subspace of the space X*. It is easily seen that *if $\omega(0) = \infty$, then $\omega(x) = \infty$ for all $x \in X$*. Indeed, assuming $\omega(x_0) < \infty$ for a certain $x_0 \in X$ we should have by iii. $\omega(-tx_0) < \infty$, where t is a positive number. Hence

$$\omega(0) = \omega\left(\frac{t}{t+1} x_0 - \frac{1}{t+1} tx_0\right) \leqslant \frac{t}{t+1} \omega(x_0) + \frac{1}{t+1} \omega(-tx_0) < \infty.$$

2. *If $0 \leqslant \omega(0) < \infty$, then the functional*

$$(*) \qquad p(x) = \inf_{t>0} \frac{\omega(tx)}{t}$$

is a Banach functional over X_*, i.e. $-\infty < p(x) < \infty$, $p(x+y) \leqslant p(x) + p(y)$ and $p(tx) = tp(x)$ for $t \geqslant 0$.

The positive homogeneity of $p(x)$ being obvious we first prove the finiteness of $p(x)$. Evidently, $p(x) < \infty$. In order to prove $p(x) > -\infty$ let us note that if $\omega(0) = 0$, then $\omega(tx)/t$ is for any $x \in X_*$ a monotone function of $t > 0$, finite in a neighbourhood of 0; hence $\omega(tx) = \int_0^t \varphi(u)\,du \geqslant$ $\geqslant t\varphi(0)$, i. e. $\omega(tx)/t \geqslant \varphi(0)$ for sufficiently small t, $\varphi(t)$ being a non--decreasing function. Thus, $\underline{\lim}_{t\to 0+} \omega(tx)/t \geqslant \varphi(0)$. Now, let $\omega(0) > 0$. Then, given $\varepsilon > 0$ and $x \in X_*$, $\omega(tx) > \varepsilon$ for sufficiently small t and $\lim_{t\to 0+} \omega(tx)/t = \infty$. But ii implies $\omega(x) + (1/t - 1)\,\omega(0) \leqslant \omega(tx)/t$ for $t \geqslant 1$. Hence, $p(x) =$ $= \inf_{t>0} \omega(tx)/t > -\infty$ *). Now, we prove $p(x)$ to be subadditive.

Given $x, y \in X_*$ and an $\varepsilon > 0$ we choose numbers $\tau, \sigma > 0$ satisfying the inequalities

$$\frac{\omega(\tau x)}{\tau} < p(x) + \frac{\varepsilon}{2} \quad \text{and} \quad \frac{\omega(\sigma y)}{\sigma} < p(y) + \frac{\varepsilon}{2}.$$

Then

$$\omega\left[\frac{\tau\sigma}{\tau+\sigma}(x+y)\right] = \omega\left(\frac{\sigma}{\tau+\sigma}\tau x + \frac{\tau}{\tau+\sigma}\sigma y\right) \leqslant \frac{\sigma}{\tau+\sigma}\omega(\tau x) + \frac{\tau}{\tau+\sigma}\omega(\sigma y),$$

whence

$$p(x+y) \leqslant \frac{\omega\left[\dfrac{\tau\sigma}{\tau+\sigma}(x+y)\right]}{\tau\sigma/(\tau+\sigma)} \leqslant \frac{\omega(\tau x)}{\tau} + \frac{\omega(\sigma y)}{\sigma} < p(x) + p(y) + \varepsilon.$$

Remark. If $\omega(x)$ is also symmetric, i. e. $\omega(-x) = \omega(x)$ and if $\omega(tx) = 0$ for all t is equivalent to $x = 0$, then $\inf_{t>0} |1 + \omega(tx)|\, t^{-1}$ is a B-norm in X_* called the *Amemiya norm*.

3. *Let us assume that a functional $\omega(x)$ satisfying the conditions* i—iii *is defined on a linear space X. Let x_t be a mapping $T \to X_T$ of a non-empty abstract set T onto $X_T \subset X$ and let β_t be a function defined on T with real values satisfying one of the following two conditions:*

(α) $X_T \subset X_*$ *and* $\omega(0) < \infty$,

(β) *if $t' \neq t''$, then $x_{t'} \neq x_{t''}$; moreover, the set X_T is linear and the functional $\beta(x_t) = \beta_t$ is additive and homogeneous over X_T.*

Then there exists over X an additive and homogenous functional $\xi(x)$ satisfying the conditions

(a) $\xi(x) \leqslant \omega(x)$ *for every* $x \in X$,

(b) $\beta_t \leqslant \xi(x_t)$ *for every* $t \in T$,

*) Let us note that if $\omega(0) < 0$, then $p(x) = -\infty$ for all $x \neq 0$; indeed, given $\varepsilon > 0$ and $x \in X_*$, $x \neq 0$, continuity of $\omega(tx)$ at $t = 0$ implies $\omega(tx)/t < -\varepsilon/t$ for t sufficiently small.

if and only if for arbitrary numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \geqslant 0$ *and arbitrary* $t_1, t_2, \ldots, t_n \in T$ *the inequality*

$$\text{(c)} \quad \sum_{i=1}^{n} \lambda_i \beta_{t_i} \leqslant \omega\left(\sum_{i=1}^{n} \lambda_i x_{t_i}\right)$$

holds.

The necessity of condition (c) being evident, we prove the sufficiency only. Let us assume $\omega(0) < \infty$; (c) with $\lambda_i = 0$, $i = 1, 2, \ldots, n$, implies $\omega(0) \geqslant 0$. If (c) is satisfied and if $p(x)$ is defined by the formula (*) for any $x \in X_*$, then

$$\text{(c}'\text{)} \quad \sum_{i=1}^{n} \lambda_i \beta_{t_i} \leqslant p\left(\sum_{i=1}^{n} \lambda_i x_{t_i}\right)$$

also holds for arbitrary $\lambda_1, \lambda, \ldots, \lambda_n \geqslant 0$, $t_1, t_2, \ldots, t_n \in T$, $x_{t_1}, x_{t_2}, \ldots, x_{t_n} \in X_*$; indeed, given any $\lambda > 0$, we have

$$\sum_{i=1}^{n} \lambda_i x_{t_i} \leqslant \frac{1}{\lambda} \omega\left(\lambda \sum_{i=1}^{n} \lambda_i x_{t_i}\right)$$

and (c′) follows. Applying the Mazur-Orlicz theorem [1] to the space X_* and to the Banach functional $p(x)$, we define an additive and homogenous functional $\xi(x)$ over X_* such that $\xi(x) \leqslant p(x)$ (whence $\xi(x) \leqslant \omega(x)$ for any $x \in X_*$ and $\beta_t \leqslant \xi(x_t)$ for all $t \in T$, $x_t \in X_*$. Thus, the conditions (a) and (b) restricted to X_* are satisfied.

Now, assuming (α), the functional $\xi(x)$ may be extended to the whole space X arbitrarily. Let us assume (β) and $\omega(0) < \infty$. Of course, we have then $\xi(x_t) = \beta_t$ for $x_t \in X_* \cap X_T$, the functional being defined above; thus, $\xi(x) = \beta(x)$ in $X_* \cap X_T$. Denote by Y_* the linear span of the set $X_* \cup Y_T$; any $y \in Y$ may written in the form $y = x + x_t$, where $x \in X_*$ and $x_t \in X_T$. It is easily seen that $y = x' + x_t' = x'' + x_t''$, where $x', x'' \in X_*$, $x_t', x_t'' \in X_T$, implies $\xi(x') + \beta(x_t') = \xi(x'') + \beta(x_t'')$, for $x' - x'' = x_t'' - x_t' \in X_* \cap X_T$ and so $\xi(x') - \xi(x'') = \beta(x_t'') - \beta(x_t')$. Thus, we may define a functional $\eta(y) = \xi(x) + \beta(x_t)$ with $x \in X_*$, $x_t \in X_T$ for any $y \in Y_*$. Obviously, the functional η is additive and homogenous and $\eta(y) = \xi(y)$ for $y \in X_*$ and $\eta(y) = \beta(y)$ for $y \in X_T$. Extending the functional η to the whole space X arbitrarily, we obtain an additive and homogenous functional satisfying (a) and (b). The case (β), $\omega(0) = \infty$ is trivial.

R e m a r k. Let us note that although the necessity of (c) in Theorem 3 holds without (α) or (β), too, the sufficiency may be false without assumption (α) or (β); as a counter-example one may take for instance $X =$ the one-dimensional space, $\omega(0) = 0$, $\omega(x) = \infty$ for all $x \neq 0$, $T = 1, \frac{1}{2}, \frac{1}{3}, \ldots, x_t = t$, $\beta_t = 1/t$.

Theorem 3 (β) may be formulated also in the following from:

4. *Let a functional $\omega(x)$ satisfying the conditions* i — iii *be defined on a linear space X and let X_0 be a linear subspace of X. If $\xi_0(x)$ is an additive and homogenous functional over X_0, satisfying the inequality $\xi_0(x) \leqslant \omega(x)$ for $x \in X_0$, then there exists over X and additive and homogenous functional $\xi(x)$ such that $\xi(x) \leqslant \omega(x)$ for $x \in X$ and $\xi(x) = \xi_0(x)$ for $x \in X_0$.*

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MATHEMATICS

Criterion for Nuclearity in Terms of Approximative Dimension

by

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At the Third Conference on Functional Analysis (Moscow, January 1956) I. M. Gelfand set (see [1], p. 6) the following problem:

"Give a definition of the "speed of approximation" of infinite-dimensional spaces by finite-dimensional spaces and derive from this concept the properties of well-approximated spaces. It is highly probable that these spaces may be most useful in practical application".

It is the notion of approximative dimension introduced by A. N. Kolmogorov ([2], [3]) that plays a fundamental role in our description of "speed of approximation". (It should be observed that a similar characterization of metric linear spaces has been proposed by A. Pełczyński [4]). On the other hand, after the papers of A. Grothendieck [5] and Gelfand-Kostyuczenko [12] it becoms clear how the nuclear spaces are important (see Def. 2 below) and how useful they are in practical application.

In the present paper the problem of Gelfand is essentially solved: for a wide class of linear topological spaces the necessary and sufficient condition for spaces to be nuclear is given in terms of Kolmogorov's notion of the approximative dimension.

Lemmas 2 and 2a are due to B. Mitiagin who also established Theorem 2a. The proof to be given here is different from that obtained previously.

1. Let E be a linear locally convex space, V be a neighbourhood *) in E, and p_V — the corresponding prenorm.

Definition 1. The set $K \subset E$ is called nuclear with respect to V, if there is a sequence $\{x_n\} \subset E$, $\sum p_V(x_n) < \infty$, $n = 1, 2, ...$, such that K is contained in the closure, with respect to the prenorm p_V, of the absolute convex hull of the sequence $\{x_n\}$.

*) In this paper we write "neighbourhood" instead of "neighbourhood of zero".

Definition 2 *). The space E is called *nuclear*, if for any neighbourhood U there is a neighbourhood V, nuclear with respect to U.

Definition 3 (see [3]). For any set $A \subset E$ and any neighbourhood U let us write

$$N(A, U; \varepsilon) = \inf \{N : A \subset \bigcup_{k=1}^{N} (x_k + \varepsilon U)\}.$$

Definition 4 ([8], p. 22). Let us call *order* of set $A \subset E$ with respect to the neighbourhood U the limit

$$\varrho(A, U) = \overline{\lim_{\varepsilon \to 0}} \frac{\log \log N(A, U; \varepsilon)}{\log 1/\varepsilon}. \tag{1}$$

Lemma 1. *Let V_i, $i = 1, 2, \ldots, n$, be neighbourhoods in E, and let $\varrho_{ik} = \varrho(V_i, V_k)$. Then* $\frac{1}{\varrho_{n1}} \geqslant \sum_{k=1}^{n-1} \frac{1}{\varrho_{k+1, k}}$.

It is sufficient to prove this proposition for $n = 3$; in this case it follows from the inequality

$$N(V_3, V_1; \varepsilon) \leqslant N(V_3, V_2; \alpha)\, N\left(V_2, V_1; \frac{\varepsilon}{\alpha}\right),$$

which is true for all $\varepsilon, \alpha < 0$, taking $\alpha = \varepsilon \frac{\varrho_{21}}{\varrho_{32} + \varrho_{21}}$.

2. Let B be a Banach space with a Schauder basis $\{e_n\}$, i.e. for any element $x \in B$ there is a unique expansion

$$x = \sum_{n=1}^{\infty} e_n'(x)\, e_n;$$

we may assume (see [9]) that

$$|e_n| = 1, \quad |e_n'| = 1 \tag{2}$$

without loss of generality.

Let $a_n \geqslant 1$, $a_n \to \infty$, be a number sequence with a convergence exponent α, i.e. $\alpha \doteq \inf\left\{\gamma; \sum_n \left(\frac{1}{a_n}\right)^{\gamma} < \infty\right\}$.

Let us consider the cube $\pi = \{x \in B; |a_n e_n'(x)| \leqslant 1\}$ and the octahedron $\sigma = \{x \in B : \sum |a_n e_n'(x)| \leqslant 1\}$.

Lemma 2. *For Banach space B with a Schauder basis the following inequalities (the right sides with positive denominators) are true*:

$$\alpha \leqslant \varrho(\pi, S) \leqslant \frac{\alpha}{1 - \alpha}$$

$$\varrho(\sigma, S) \leqslant \alpha \leqslant \frac{\varrho(\sigma, S)}{1 - \varrho(\sigma, S)};$$

where S is the unit sphere in B.

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*) Of the many equivalent definitions of this notion we use the definition of nuclearity, the most convenient for our present purpose (cf. [5] — [7]).

First let us prove a similar proposition for ellipsoids in the spaces l^p.

LEMMA 2a. *In the space l^p, $(0 < p < \infty)$, an order ϱ of the ellipsoid*

$$\vartheta = \left\{ \xi : \left(\sum_n |a_n \xi_n|^p \right)^{1/p} \leq 1 \right\}$$

is equal to the convergence exponent a of the sequence $\{a_n\}$.

Proof. In the k-dimensional space R^k generated by the first k vectors of the basis the ratio of volumes

$$(3) \qquad \frac{\mathrm{Vol}(\vartheta \cap R^k)}{\mathrm{Vol}(\vartheta \cap R^k)} \quad \text{is equal to} \quad \prod_{n=1}^{k} a_n^{-1}.$$

Using this remark and repeating essentially considerations of par. 6 of [8] (Theorem 16), we may get the following inequalities

$$(4) \qquad K_p^{m(2/\varepsilon)} \prod_{\varepsilon a_k < 2} \frac{2}{\varepsilon a_k} \geq N(\vartheta, S; \varepsilon) \geq \prod_{\varepsilon a_k < 1} \frac{1}{\varepsilon a_k},$$

where $m(t)$ is the number of elements of the sequence $\{a_n\}$ lying in the interval $(0, t)$ and K_p is a constant which does not depend on ε.

It is known (see e.g. [10], p. 20) that

$$(5) \qquad a = \overline{\lim_{t \to \infty}} \frac{\log m(t)}{\log t}.$$

From (4) it follows:

$$\log N(\vartheta, S; \varepsilon) \geq \log \prod_{\varepsilon a_k < 1} \frac{1}{\varepsilon a_k} = \sum_{\varepsilon a_k < 1} \log \frac{1}{\varepsilon a_k} = \int_0^{1/\varepsilon} \log \frac{1}{\varepsilon t}\, dm(t) =$$

$$m(t) \log \frac{1}{\varepsilon t} \Big|_0^{1/\varepsilon} + \int_0^{1/\varepsilon} \frac{m(t)}{t}\, dt \geq m\left(\frac{1}{\varepsilon e}\right) \int_{1/(\varepsilon e)}^{1/\varepsilon} \frac{dt}{t} = m\left(\frac{1}{\varepsilon e}\right).$$

Similarly estimating from above the logarithm of the left side of inequality (4) we get:

$$m\left(\frac{2}{\varepsilon}\right)\left(\log K_p + \log \frac{2}{\varepsilon}\right) \geq \log N(\vartheta, S; \varepsilon) \geq m\left(\frac{1}{\varepsilon e}\right),$$

where K_p does not depend on ε. The statement of Lemma 2a and formulae (1) and (5) follow immediately from these inequalities.

Proof of Lemma 2. In virtue of (2) in every R^n the unit cube $\pi_1^n = \{x \in R^n : |e_k'(x)| \leq 1, \quad k = 1, \ldots, n\}$, the unit octahedron $\sigma_1^n = \{x \in R^n : \sum_1^n |e_k'(x)| \leq 1\}$ and the section $S^n = S \cap R^n$ are connected by the inclu-

sions $\pi_1^n \supset S^n \supset \sigma_1^n$. In this general case instead of the exact equality (3) we have the relations

$$\mathrm{Vol}(\pi_1^n) \geqslant \mathrm{Vol}(S^n) \geqslant \mathrm{Vol}(\sigma_1^n),$$

$$\frac{\mathrm{Vol}(\pi \cap R^n)}{\mathrm{Vol}(\pi_1^n)} = \frac{\mathrm{Vol}(\sigma \cap R^n)}{\mathrm{Vol}(\sigma_1^n)} = \prod_1^n a_k^{-1}, \qquad \frac{\mathrm{Vol}(\pi_1^n)}{\mathrm{Vol}(\sigma_1^n)} = n!$$

and instead of (4) in a similar way we get the inequalities

$$C^{m(2/\varepsilon)} \prod_{\varepsilon a_k < 2} \frac{1}{\varepsilon a_k} \geqslant N(\sigma, S; \varepsilon) \geqslant \prod_{k \varepsilon a_k < 1} \frac{1}{\varepsilon a_k k},$$

$$C_\delta^{m_\delta(2 \cdot \varepsilon)} \prod_{\varepsilon a_k < 2} \frac{2 k^{1+\delta}}{\varepsilon a_k} \geqslant N(\pi, S; \varepsilon) \geqslant \prod_{\varepsilon a_k < 1} \frac{1}{\varepsilon a_k},$$

where $m_\delta(t)$ is the function of the sequence $a_k/k^{1+\delta}$. Introducing the function $l(t)$ of the sequence $a_k k$ and making use of the fact that if $\Sigma \gamma_k < \infty$, $\gamma_k \downarrow 0$, then $n\gamma_n \leqslant C$ it is not difficult, by means of these inequalities, to get the statement of Lemma 2.

It is to be noticed that all inequalities of Lemma 2 are exact, i.e. they cannot be improved as it is shown by the examples π and σ in the spaces l^1 and c_0.

COROLLARY. *If $\varrho(\sigma, S) < \frac{1}{3}$ then π is nuclear with respect to S.*

Indeed in this case $\alpha < \frac{1}{2}$ and if δ is sufficiently small, then the cube π is contained in the closure of the absolute convex hull of the sequence $\{n^{1+\delta} a_n^{-1} e_n\}$, and $\sum_n n^{1+\delta} a_n^{-1} < \infty$.

3. THEOREM 1. *In the nuclear space E for every bounded set A and any neighbourhood U the following relation is true*

$$\varrho(A, U) = 0.$$

In fact, in the nuclear space there is (see [5]) a fundamental system of prenorms p generated by the inner products $(x, y)_p$. Therefore, for any neighbourhood $V \subset U$ generating a structure of Hilbert space H_V in E there is a neighbourhood W such that W is an ellipsoid in H_W with a convergent series of semiaxes. Then Lemma 2a (for $p = 2$) gives the inequality $\varrho(W, V) \leqslant 1$. As $\varrho(A, U)$ $\varrho(W, V)$ the proof may be completed, using Lemma 1.

LEMMA 3*). *Let E be a metric space with fundamental system of neighbourhoods $\{U_K\}$, and*

$$\varrho(K, U) < C < \infty$$

for all compacts K and all neighbourhoods U in E. Then for any neighbourhood V there is a neighbourhood W such that $\varrho(W, V) < 2C$.

*) This Lemma is a special case of the Theorem of C. Bessaga and A. Pełczyński (see [11], Th. 1).

Proof. If for some neighbourhood $\varrho(U_k, U) > 2C$ for all k then for every k (in virtue of Theorem 4, [8]) there exist such finite sets $F_k \subset U_k$ and $\delta_k > 0$, $\delta_k \downarrow 0$, that $\log N(F_k, U; \delta_k) > (1/2\delta_k)^c$. But then the set $F = \bigcup_{k=1}^{\infty} F_k \cup \{0\}$ is compact in E and $\varrho(F, U) > C$.

This contradiction proves Lemma 3.

THEOREM 2 *). *Let E be a locally convex space such that*

$$\varrho(K, U) > C < \infty$$

for all compacts K and neighbourhoods U in E.

If E is

a) countable-Hilbert space, or

b) a complete metric space with Schauder basis **), *or*

c) a strong dual of a space of type a) or b) — then E is a nuclear space.

Proof. If the condition a) holds, the statement of the Theorem follows from Lemmas 3 and 1 and Corollary.

If the condition b) holds, we may define the topology of the space E by such a system of prenorms that in completion of E with respect to each of these prenorms the vectors $\{e_k\}$ generates a Schauder basis (cf. [13], p. 380). Just as in case a) the Theorem follows from Lemmas 3 and 1 and Corollary.

The case c) may be derived by virtue of duality.

An analysis of the proof of case b) of Theorem 2 also shows that the following theorem holds.

THEOREM 3. *In nuclear metric space every Schauder basis is an unconditional basis.*

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*) As shown in the paper of B. Mitiagin [14], the statement of Theorem 2 holds for any metric space.

**) For a very special case of spaces of type b) Theorems 1 and 2 have been announced by S. Rolewicz [11].

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THEORETICAL PHYSICS

Polarization of Diffracted Electromagnetic Wave

by

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Presented by W. RUBINOWICZ on April 29, 1960

Notwithstanding its well known drawbacks, Kirchoff's theory accounts correctly for the intensity of the light diffracted in the neighbourhood of the boundary of geometrical shadow. This decides in favour of Kirchoff's theory for solving particular diffraction problems. Kirchoff's assumptions, however, are insufficient for yielding the dependence of the polarization of the diffracted wave on that of the incident wave, because his theory is a scalar one. From the few hitherto available exact solutions of diffraction problems, the polarization of the diffracted wave may be said to be of the same kind as that of the incident wave. However, in considering diffraction on an aperture Σ of arbitrary form, there are, in general, no grounds for such a statement. Information on the polarization problem is to be obtained from approximate electromagnetic theories; among these, Kottler's [1] theory should be mentioned in the first place. By adapting this theory to Fraunhofers diffraction, data on polarization of the diffracted wave can be obtained.

The present paper brings proof that polarization of the diffracted wave is of the same kind as that of the incident wave. E.g., if the incident wave is plane polarized, the wave diffracted at any angle is plane polarized too. The theorem holds also for either of the remaining kinds of polarization, i.e. circular and elliptic.

This fact throws some additional light on Kottler's approximate electromagnetic theory. Thus, if experimental data demonstrated, even in part, that polarization of the diffracted wave was of the same kind as that of the incident wave, such data would weigh in favour of Kottler's ideas.

1. On the assumption of Kottler's theory [1], the present author derived [2] formulae yielding the electromagnetic diffracted field for the case of Fraunhofers diffraction. These (paper [2], formulae (9a, b)) describe the electric and magnetic field intensities of the diffracted wave and hold for any direction of incidence of the plane polarized wave upon the aperture Σ.

Assume the aperture Σ to be cut out in an infinite plane screen situated in the xy plane. The z-direction of the reference system assumed coincides with that of the versor $\vec{n}$ normal to Σ. Coming from the half-space $z<0$, a plane wave is incident on Σ, its direction of propagation being given by the versor $\vec{p}$. This direction of the diffracted wave is given by the versor $-\vec{a}$. The relationships are those of Fig. 1.

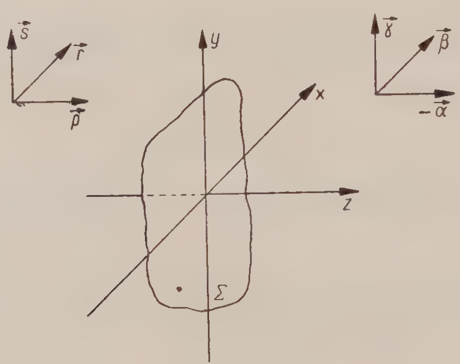

Fig. 1

Eqs. (9a, b) of paper [2] will now be rewritten as follows:

$$\vec{E}^F = \frac{ik}{4\pi}\{-\vec{A}(\vec{a}\vec{F})+\vec{A}(\vec{p}\vec{F})-\vec{p}(\vec{A}\vec{F})+\vec{F}(\vec{A}\vec{a})+\vec{a}(\vec{p}\vec{a})(\vec{A}\vec{F})-\vec{a}(\vec{A}\vec{a})(\vec{F}\vec{p}\}, \tag{1a}$$

$$\vec{H}^F = \frac{ik}{4\pi}\{-(\vec{p}\times\vec{A})(\vec{a}\vec{F})+(\vec{F}\times\vec{A})+\vec{F}[(\vec{p}\times\vec{A})\vec{a}]-\vec{a}[(\vec{A}\times\vec{a})\vec{F}]\}. \tag{1b}$$

Here $\vec{A}$ denotes the vector-amplitude of the incident wave:

$$\vec{F} = \int_{\Sigma} \vec{n}\exp[-ik\vec{R}(\vec{a}+\vec{p})]\,d\Sigma,$$

$\vec{R}$ denoting the radius vector. The computations to be carried out are greatly simplified by the fact that the vector $\vec{F}$ has been put in the above form. Moreover, the following relationships are readily verified:

$$-\vec{a}\vec{E}^F = 0, \qquad -\vec{a}\vec{E}^F = \vec{H}^F.$$

Hence, as required, the diffracted electromagnetic waves in the case of Fraunhofer's diffraction are seen to possess the structure of a plane wave. (More exactly, Eqs. (1 a, b) describe the amplitudes of the diffracted wave).

2. In order to prepare for a discussion on the polarization of the diffracted wave and of its possible relationships with that of the incident wave, two co-ordinate systems will be introduced: one wherein the polarization of the incident wave will be arbitrarily fixed (a right-hand system of the vectors $\vec{r}, \vec{s}, \vec{p}$) and another for investigating the polarization of the diffracted wave (a right-hand system of the versors $\vec{\beta}, \vec{\gamma}, -\vec{a}$ (Fig. 1)).

Assume now that a plane wave, determined in the co-odinate system $\vec{r}, \vec{s}, \vec{p}$ as follows:

(2a) $$\vec{E}' = \vec{E}_0 e^{-i\vec{R}\vec{p}K}, \qquad \vec{E}'_0 = [E'_r, E'_s, 0],$$

(2b) $$\vec{H}' = \vec{H}'_0 e^{-i\vec{R}\vec{p}K}, \qquad \vec{H}'_0 = [-E'_s, E'_r, 0],$$

is incident on the aperture Σ. The factor e^{iwt} accounting for the time dependence of the vectors will be omitted everywhere. The amplitudes $\vec{E}'_0, \vec{H}'_0$ are, in general, complex vectors.

In the system. x, y, z the versors $\vec{r}, \vec{s}, \vec{p}$ present the following components:

$$\vec{r} = \left[\frac{p_3}{\sqrt{1-p_2^2}}, \quad 0, \quad \frac{-p_1}{\sqrt{1-p_2^2}}\right],$$

$$\vec{s} = \left[\frac{-p_1 p_2}{\sqrt{1-p_2^2}}, \sqrt{1-p_2^2}, \frac{-p_2 p_3}{\sqrt{1-p_2^2}}\right],$$

$$\vec{p} = [p_1, p_2, p_3].$$

If the versor $\vec{p}$ is rewritten in x, y, z, one of the components of $\vec{r}$ or $\vec{s}$ can always be arbitrarily chosen. Here, the second component of the vector $\vec{r}$ has been assumed to vanish in the x, y, z co-ordinate system ($r_2 = 0$). Transforming the plane wave of Eqs. (2a, b) (or rather the amplitudes $\vec{E}_0, \vec{H}'_0$ to the system x, y, z we get

(3a) $$\begin{cases} E_{ox} = A_x = E'_r \dfrac{p_3}{\sqrt{1-p_2^2}} - E'_s \dfrac{p_1 p_2}{\sqrt{1-p_2^2}}, \\ E_{oy} = A_y = E'_s \sqrt{1-p_2^2}, \\ E_{oz} = A_z = -E'_r \dfrac{p_1}{\sqrt{1-p_2^2}} - E'_s \dfrac{p_2 p_3}{\sqrt{1-p_2^2}}, \end{cases}$$

(3b) $$\begin{cases} H_{ox} = (\vec{p} \times \vec{A})_x = -E'_r \dfrac{p_1 p_2}{\sqrt{1-p_2^2}} - E'_s \dfrac{p_3}{\sqrt{1-p_2^2}}, \\ H_{oy} = (\vec{p} \times \vec{A})_y = E'_r \sqrt{1-p_2^2}, \\ H_{oz} = (\vec{p} \times \vec{A})_z = -E'_r \dfrac{p_2 p_3}{\sqrt{1-p_2^2}} + E'_s \dfrac{p_1}{\sqrt{1-p_2^2}}, \end{cases}$$

where $E_{ox}, \ldots$ etc. denote the amplitudes of the incident wave in the co-ordinate system x, y, z, respectively.

In the x, y, z co-ordinate system the versors $\vec{\beta}, \vec{\alpha}, -\vec{\gamma}$, have the following components:

$$\vec{\beta}=\left[\frac{-\alpha_3}{\sqrt{1-\alpha_2^2}},\quad 0,\quad \frac{\alpha_1}{\sqrt{1-\alpha_2^2}}\right],$$

$$\vec{\gamma}=\left[\frac{-\alpha_1\alpha_2}{\sqrt{1-\alpha_2^2}},\sqrt{1-\alpha_2^2},\frac{-\alpha_2\alpha_3}{\sqrt{1-\alpha_2^2}}\right],$$

$$-\vec{\alpha}=[-\alpha_1,-\alpha_2,-\alpha_3].$$

In the system $\vec{\beta},\vec{\gamma},-\vec{\alpha}$, the vector $\vec{E}^F$ of Eq. (1a) has the following components:

$$\text{(4a)}\qquad \begin{cases} E_{-\alpha}=0, \\ E_\beta=\dfrac{ik}{4\pi}\{(\vec{A}\vec{\beta})(\vec{p}\vec{F}-\vec{\alpha}\vec{F})-(\vec{\beta}\vec{p})(\vec{A}\vec{F})+(\vec{F}\vec{\beta})(\vec{A}\vec{\alpha})\}, \end{cases}$$

$$\text{(4b)}\qquad E_\gamma=\frac{ik}{4\pi}\{(\vec{A}\vec{\gamma})(\vec{p}\vec{F}-\vec{\alpha}\vec{F})-(\vec{\gamma}\vec{p})(\vec{A}\vec{F})+(\vec{F}\vec{\gamma})(\vec{A}\vec{\alpha})\}.$$

The values of all scalar products in Eqs. (4 a, b) are known in the co-ordinate system x, y, z.

3. We now proceed to investigate the polarization of the diffracted wave as dependent on that of the incident wave. With this aim, the amplitudes E'_r, E'_s of the incident wave will be assumed to be of the form

$$\text{(5)}\qquad \begin{cases} E'_r=E'_{or}\,e^{i\delta_1}, \\ E'_s=E'_{os}\,e^{i(\delta_1+\delta)}, \end{cases}$$

where E'_{os} and E_{or} are real quantities, and δ_1 and $\delta_1+\delta$ the respective phases. As a rule, the incident wave is assumed to be elliptically polarized, the difference in phase being δ.

Taking into account the formula for $\vec{F}$ Eqs. (3a), for the vector $\vec{A}$, Eqs. (4 a, b) and Eq. (5), we have

$$\text{(6)}\qquad \begin{cases} E_\beta=\dfrac{ik}{4\pi}Fe^{i\delta_1}(B+C\cos\delta+iC\sin\delta), \\ E_\gamma=\dfrac{ik}{4\pi}Fe^{i\delta_1}(D+G\cos\delta+iG\sin\delta), \end{cases}$$

wherein, for brevity, the following notations have been used:

$$\text{(7)}\qquad \begin{cases} B=E'_{or}\beta_1\sqrt{1-p_2^2}+E'_{or}\dfrac{\gamma_2 p_3}{\sqrt{1-p_2^2}}, \\ C=-E'_{os}\gamma_2\dfrac{p_1p_2}{\sqrt{1-p_2^2}}-E'_{or}\gamma_1\sqrt{1-p_2^2}, \\ D=E'_{or}\gamma_1\sqrt{1-p_2^2}+E'_{os}\gamma_2\dfrac{p_1p_2}{\sqrt{1-p_2^2}}, \\ G=E'_{os}\gamma_2\dfrac{p_3}{\sqrt{1-p_2^2}}+E'_{os}\beta_1\sqrt{1-p_2^2}. \end{cases}$$

Putting $\frac{k}{4\pi} F e^{i(\delta_1+\pi/2)} = He^{i\varphi}$, Eq. (6) yields

(8a) $$|E_\beta| = H\sqrt{B^2+2BC\cos\delta+C^2},$$

(8b) $$|E_\gamma| = H\sqrt{D^2+2DG\cos\delta+G^2}.$$

Denoting the difference of the arguments $(\varphi_\beta+\varphi)-(\varphi_\gamma+\varphi)$ of the complex quantities E_β, E_γ by ψ, we have

(9) $$\operatorname{tg}\psi = \frac{(BG-CD)\sin\delta}{BD+CG+(CD+BG)\cos\delta}.$$

The following important conclusion can be derived from Eq. (9): if the wave incident upon the aperture Σ is polarized elliptically, the wave diffracted (at any angle) is elliptically polarized, too; in general, $\psi \neq \delta$.

It will now be assumed that the incident wave is circularly polarized, thus

$$E_{or} = E'_{os},$$

(10) $$\delta = m\frac{\pi}{2}, \quad m = \pm 1, \pm 3, \ldots$$

In the present case, Eqs. (7)—(10) yield

(11) $$|E_\beta| = |E_\gamma|, \quad \operatorname{tg}\psi = \pm\infty,$$

i.e.

(12) $$\psi = m\frac{\pi}{2}, \quad m = \pm 1, \pm 3, \ldots$$

Hence, the wave diffracted at any angle is circularly polarized, too.

Finally, the incident wave will be assumed to be plane polarized, i.e. $\delta = m\pi$, with $m = 0, \pm 1, \pm 2, \ldots$ In the present case, by Eq. (9), $\operatorname{tg}\psi = 0$, or

(13) $$\psi = m\pi, \quad m = 0, \pm 1, \pm 2, \ldots$$

Thus, the wave diffracted at an arbitrary angle is plane polarized too.

Concluding, the polarization of a wave diffracted on an arbitrary aperture Σ cut out in an infinite plane screen is of the same kind as that of the incident wave. Thus, within the framework of Kottler's theory and for Fraunhofer's diffraction the law of conservation of polarization species holds.

4. The simple example of a plane polarized, plane wave ($\vec{p} = [0, 0, 1]$) perpendicularly incident upon an aperture cut out in a plane screen will be considered. The incident wave is given by

(14) $$\vec{E}_0 = \vec{A}e^{-iK\vec{R}\vec{p}}, \quad \text{with} \quad \vec{A} = [A_x, A_y(-1)^m, 0].$$

A_x and A_y are real quantities. By Eq. (1a),

(15) $$\vec{E}^F = \frac{ik}{4\pi} F[(1-\alpha_3)\vec{A} - \vec{\alpha}(\vec{A}\vec{\alpha}) + \vec{\varkappa}(\vec{A}\vec{\alpha})],$$

with $\vec{\varkappa}$ being the unit vector in the z-direction. The incident wave of Eq. (14) is plane polarized by assumption. The diffracted wave given by Eq. (15) is, of course, also plane polarized. The angle ϱ between the planes of polarization of the diffracted and incident waves will now be computed.

This amounts to computing the quantity

$$\cos\varrho = \frac{Re(\vec{E}_0)\,Re(\vec{E}^F)}{|Re\,\vec{E}_0|\cdot|Re\,\vec{E}^F|}.$$

By Eqs. (14) and (15), the result

$$\cos\varrho = \pm\left(1 - \frac{\cos^2\chi}{1-\alpha_3}\right), \tag{16}$$

is readily obtained; χ denotes the angle subtended by the vector perpendicular to the plane of polarization of the incident wave and the vector $\vec{\alpha}$. Clearly, $\cos^2\chi$ is a measurable quantity. It should be stressed that $\cos\delta$ is independent of the shape of the aperture Σ, as $\cos^2\chi$ presents no such dependence.

In particular, if the incident wave is plane polarized along the y-axis. i.e. if $\vec{A} = [0, A, 0]$, and if diffracted waves propagating in the direction $(0, -\alpha_2, \alpha_3)$ are considered, then $\cos^2\chi = \alpha_2^2$, and

$$\cos\varrho = \pm\alpha_3. \tag{17}$$

If the wave $\vec{E}^F$ propagating in the direction $(0, 0, -\alpha_3 = 1)$ of the incident wave is considered, then $\cos\varrho = \pm 1$, i.e. $\varrho = 0$ or π, and thus, in the case under consideration, the plane of polarization undergoes no rotation. In general, $\cos\varrho$ will depend on the components of the vector $\vec{p}$.

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THEORETICAL PHYSICS

New Possible Phenomenon in Antiferromagnetics

by

H. COFTA

Presented by W. RUBINOWICZ on June 7, 1960

Introduction

In one of our previous notes [1] we have pointed out that in certain antiferromagnetic superstructures a considerable anisotropy of dispersion of spin waves must occur, caused exclusively by the superstructure. We have proposed for this effect the name "superstructural anisotropy of spin wave dispersion" [2]. In the present paper we give the results of exact calculation of dispersions laws for these interesting cases. These calculations have been carried out in connection with some geometrical considerations (par. 1). The results obtained confirm entirely our suggestions presented in par. 1. In contradiction to isotropic propagation, assumed in all existing calculations, the spin waves appear to be governed by an anisotropic dispersion law in most antiferromagnetic superstructures. A point of peculiar interest results from taking into account the next--nearest interactions: measurement of superstructural anisotropy parameters furnishes immediately the ratio $\eta = J_1/J_2$ of nearest interactions to the next-nearest ones.

Geometrical considerations

A very useful auxiliary method in analyzing the superstructural anisotropy of spin waves propagation consists in a proper systematization of superstructures. However, the classification introduced previously [3] must be developed in a more precise way [4]. Let us consider an antiferromagnetic lattice as composed of two identical interpenetrating ferromagnetic sublattices A and B with opposite spins. Such an order in which each lattice point is a center of symmetry will be called a "symmetric superstructure", whereas the term "nonsymmetric" will be assigned to all other cases. A very interesting class of symmetric superstructures is formed by the "regular superstructures" in which each of both sublattices A and B has a translational character. In this case there occur sets of alternatingly arranged (... $ABAB$ ...) parallel ferromagnetic planes (F-pla-

nes). This fact allows to formulate suggestions concerning the behaviour of spin waves in regular superstructures. Let us confine ourselves to cubic lattices where purely structural effects do not appear. With exception of natural orders [3] (where all the nearest neighbours of a lattice point have opposite spins) all regular superstructures should show an anisotropy of spin wave dispersion, the preferred direction being perpendicular to the unique set of F-planes. More detailed predictions of such anisotropic effects are also possible [4] and a detailed analysis of these questions will be presented in Acta Physica Polonica [5]. Let us only remark, that an analogous consideration of the order of substructures (as consisting of second neighbours solely) allows to predict the character of next-nearest interactions term in the dispersion law.

Theoretical treatment

In order to justify our predictions mentioned above we can start from any general dispersion law for spin waves in antiferromagnetics, derived in oscillator approximation (i.e. by neglecting the interactions between spin waves). E.g. we may use the formula (24) of Ziman [6] obtained in the Holstein-Primakoff approximation (for the range of validity of Ziman's formula, see [3] or [4]) or the identical formula (25) by Cofta [3] obtained in the semiclassical approximation.

Practically we are dealing always with long spin waves, for which the quoted formulae yield the following expression for the square of energy:

$$E^2 = 2\,S^2\,\hbar\varkappa\,\{g(\boldsymbol{k}) - f(\boldsymbol{k})\}, \tag{1}$$

where we have put for simplicity the magnetic anisotropy field $H_A = 0$. Then

$$\left|\begin{aligned} &\hbar\varkappa = 2\,(J_1 z_1 + J_2 z_2), \\ &f(\boldsymbol{k}) = J_1 \sum_1^{+} (\boldsymbol{k}\boldsymbol{r})^2 + J_2 \sum_2^{+} (\boldsymbol{k}\boldsymbol{r})^2, \\ &g(\boldsymbol{k}) = J_1 \sum_1^{-} (\boldsymbol{k}\boldsymbol{r})^2 + J_2 \sum_2^{-} (\boldsymbol{k}\boldsymbol{r})_2. \end{aligned}\right. \tag{2}$$

The following symbols have been used here:

S — the absolute value of z — component of atomic spin;

$\hbar$ — the Planck constant divided by 2π;

J_1 and J_2 — exchange integrals for nearest and next-nearest interactions respectively;

z_1 and z_2 — numbers of nearest and next-nearest neighbours, respectively with opposite spins;

$\boldsymbol{k}$ — propagation vector of spin wave;

$\boldsymbol{r}$ — vector of relative position of neighbours;

$\sum\limits_{1}^{+}$ and $\sum\limits_{1}^{-}$ — summations which should be carried out over the nearest neighbours with the same spins (+) or opposite spin (—), respectively;

$\sum\limits_{2}^{+}$ and $\sum\limits_{2}^{-}$ — analogous symbols for summations over next-nearest neighbours.

Writing Eq. (1) in the form:

$$E^2 = \boldsymbol{kMk}, \tag{3}$$

where $\boldsymbol{M}$ is the appropriate symmetric tensor, and denoting by a the lattice constant, we obtain the following results for the 5 particular superstructures in cubic lattices.

a) For the *sc* lattice with (001) F-planes:

$$\begin{cases} M_{xx} = M_{yy} = -16\, a^2 S^2 J_1 (J_1 + 4\, J_2) \\ M_{zz} = 16\, a^2 S^2 (J_1 + 4\, J_2)^2 \\ M_{xy} = M_{yz} = M_{zx} = 0. \end{cases} \tag{4}$$

An uniaxial anisotropy with preferred direction [001] is obvious. This anisotropy becomes particularly conspicuous for the next-nearest interactions ($J_1 = 0$). The nearest one ($J_2 = 0$) privilege the same direction.

b) For the *sc* lattice with (110) and ($1\bar{1}0$) *F*-planes, we get:

$$\begin{cases} M_{xx} = M_{yy} = 32\, a^2 S^2 J_1 (J_1 + 2\, J_2) \\ M_{zz} = 32\, a^2 S^2 (4\, J_2 - J_1)(J_1 + 2\, J_2) \\ M_{xy} = M_{yz} = M_{zx} = 0. \end{cases} \tag{5}$$

According to our predictions, Eq. (5) presents anisotropy with prefeι red direction [001]. The effects of next-nearest interactions have a similar character.

c) For the *bcc* lattice with (110) *F*-planes, taking new axes x', y', z' in the directions: [$1\bar{1}0$], [110], [001] respectively, we have:

$$\begin{cases} M'_{xx} = 32\, a^2 S^2 (J_2{}^2 - J_1{}^2) \\ M'_{yy} = 32\, a\, S^2 (J_1 + J_2{}^2) \\ M'_{zz} = -32\, a\, S^2 J_2 (J_1 + J_2) \\ M'_{xy} = M'_{yz} = M'_{zx} = 0. \end{cases} \tag{6}$$

Here also an unquestionable anisotropy occurs, one preferred direction being perpendicular to *F*-planes. The next-nearest interactions $J_1 = 0$ privilege the [001] direction.

d) For the *fcc* lattice with (001) *F*-planes Eq. (1) yields:

$$
\text{(7)} \qquad \begin{cases} M_{xx} = M_{yy} = -64\, a^2 S^2 J_1 J_2 \\ M_{zz} = 64\, a^2 S^2 J_1 (J_1 - J_2) \\ M_{xy} = M_{yz} = M_{zx} = 0. \end{cases}
$$

The anisotropy presented by Eq. (7) prefers also the direction [001] perpendicular to *F*-planes.

e) For the well known case of the *fcc* lattice with (111) *F*-planes we obtain, taking new axes x', y', z' in directions $[11\bar{2}]$, $[\bar{1}10]$, [111] respectively, the following results:

$$
\text{(8)} \qquad \begin{cases} M'_{xx} = M'_{yy} = C\,(J_1 + 2\,J_2) \\ M'_{zz} = 2\,C\,(J_2 - J_1) \\ M'_{xy} = M'_{yz} = M'_{zx} = 0 \end{cases}
$$

where

$$
C = 24\, a^2 S^2 (J_1 + J_2).
$$

An uniaxial anisotropy privileges markedly the direction [111] perpendicular to *F*-planes. Next-nearest interactions prefer no directions (in agreement with detailed predictions presented in [4] and [5]).

Hence the results (4) to (9) prove entirely our hypotheses as to the existence and some properties of the superstructural anisotropy of spin waves propagation connected with the presence of *F*-planes. Detailed calculations, taking into account magnetic anisotropy (done also in [4]), will be published shortly. Let us remark, that in "natural" orders more than two different sets of *F*-planes occur, so that their effects are mutually compensated (by reason of crystal symmetry) according to the well-known isotropic character of such superstructures.

Experimental possibilities

Two methods of measuring the predicted effects seem to be applicable. The first one is the method of inelastic scattering of neutrons by spin waves. This method is capable of revealing the dispersion law for spin waves [7] and its adaptation to our purposes seems to be possible (J. Janik and T. Riste, private communication). The second method, which might be suitable for superstructural anisotropy, consists in resonant excitation of spin waves in thin antiferromagnetic plates by a uniform *rf* field [8], [9], The theory of such a possible experiment is being prepared by the author.

We hope, that at least three advantages may be gained from observations of superstructural anisotropy.

1° The detection of this phenomenon should yield a supplementary confirmation of our present views upon the mechanism of spin waves and possibly some new informations as to their propagation.

2° Such investigations should make possible the determination of superstructure, when other methods give ambiguous information.

3° The measurement of superstructural anisotropy parameters should present a method of determining the ratio $\eta = J_1/J_2$. Let us discuss nearer this last question more extensively.

As we have seen, the effect of next-nearest interactions differs from that of nearest interactions. In other words the parameters of the iso-energetic surface $E = \text{const}$ in $\boldsymbol{k}$-space depend on η, which is easily seen from the formulas (4)—(9). Let us consider e.g. the case of the *fcc* lattice with the superstructure consisting in the presence of (111) F-planes. When considering only the realistic values $-1 < \eta < 1$ we get for the isoenergetic surface an ellipsoid of revolution. The ratio $a:c$ of its axes varies as

$$\sqrt{2}\,\{(1-\eta)/(2+\eta)\}^{1/2}$$

taking values between 0 and 2, which should be determined experimentally. Note, that no direct informations on η or related quantities is known hitherto. Closer analysis of this question for all regular superstructures and detailed discussion of isoenergetic surfaces may be found in [4] and will be published in subsequent papers.

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THEORETICAL PHYSICS

The Isorepresentations of Leptons

by

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Presented by L. INFELD on June 8, 1960

Introduction

The isobaric multiplet formalism, proposed by Gell-Mann [1] and Nishijima [2], was not applied to light $\frac{1}{2}$ — spin particles. Thus in this scheme all selection rules were true by assumption that for leptons $I_3 = S \equiv 0$ (S — strangeness). The isoproperties of leptons may be neglected in considerations of strong interaction. Recent search for the form of weak interactions shows, however, that determination of these isoproperties may be useful for the unified description of weak processes.

Many attempts have been made to apply the isospin assignements to leptons [3]—[10]. In these papers only the eigenvalues of I_3 are considered. Since under rotations, generated by I_3 in (1, 2-isoplane) $z_{;1}$ transforms identically with $z_{;2}$ ($z_{;\alpha}$ — 2-spinor in isospace), the question of suitable isorepresentation arises.

In this paper we introduce two schemes (A and B), for which $S = 0$. The electric charge Q is expressed by the following formula:

$$\frac{Q}{e} = I_3 + \frac{L}{2}, \tag{1.1}$$

where L — lepton number, for baryons and mesons equals zero. All leptons are grouped in two isodoublets, the fourth lepton being the second neutrino. The existence of a second neutrino was assumed by Goto [4] and investigated in detail by Lippmanow [9].

Our representations are written in 3-dimensional isospace E_3. The extension to 4-dimensional (Euclidean E_4 or Minkowski's $E_4^{(1)}$) isospaces can be simply achieved. In particular, if we extend E_3 to $E_4^{(1)}$, we obtain the supplement to Rzewuski's classification of elementary particles with respect to the full transformation group in spinor space [11].

Scheme A

It is well-known that N, Ξ doublets can be described in E_3 as follows:

$$(2.1)\qquad \begin{aligned} N:\ N^{+} &= \begin{pmatrix} F_{\alpha;\,1} \\ F^{\dot{\alpha};}{}_{1} \end{pmatrix} \qquad & N^{0} &= \begin{pmatrix} F_{\alpha;\,2} \\ F^{\dot{\alpha};}{}_{2} \end{pmatrix} \\ \Xi:\ \Xi\ &= \begin{pmatrix} D_{\alpha;\,1} \\ D^{\dot{\alpha};}{}_{1} \end{pmatrix} \qquad & \Xi' &= \begin{pmatrix} D_{\alpha;\,2} \\ D^{\dot{\alpha};}{}_{2} \end{pmatrix} \end{aligned}$$

Scheme A is obtained by the interchange

$$(2.2)\qquad N \to L_1 \qquad \Xi \to L_2\,,$$

where

$$(2.3)\qquad \begin{aligned} L_1:\ L_1^{+} &= \begin{pmatrix} e \\ u \end{pmatrix} \qquad & L_2^{0} &= \begin{pmatrix} \nu_{+} \\ \tilde{\nu} \end{pmatrix}, \\ L_2:\ L_2^{+} &= \begin{pmatrix} \mu \\ e_{-} \end{pmatrix} \qquad & L_1^{0} &= \begin{pmatrix} \tilde{\nu}_{+} \\ \nu_{-} \end{pmatrix}. \end{aligned}$$

$e_{\pm}, \mu_{\pm}, \nu_{\pm}, \tilde{\nu}_{\pm}$ denotes chirality eigenfunctions, for example

$$(2.4)\qquad e_{\pm} = \frac{1}{2}(1 \pm \gamma_5)\, e$$

and $\tilde{\nu}$ the second neutrino ($\frac{1}{2}$ — spin mass-less neutral particle).

According to (1.1) e and μ have electric charge $+e$, if lepton number L is generated as follows:

$$(2.5)\qquad \boldsymbol{L}:\ L_1' = e^{i\alpha} L_1 \qquad L_2' = e^{i\alpha} L_2\,.$$

L plays an analogous role for leptons as hypercharge for the (N, Ξ) — pair.

We introduce a second parameter H, analogous to the baryon number, by way of the following transformation:

$$(2.6)\qquad \boldsymbol{H}:\ L_1' = e^{i\beta} L_1 \qquad L_2' = e^{-i\beta} L_2\,.$$

It is easily seen from (2.3) and (2.4) that (2.6) can be expressed in the conventional Dirac's 4-spinor notation as follows:

$$(2.7)\qquad \boldsymbol{H}:\ \begin{aligned} e' &= e^{i\beta\gamma_5} e \qquad & \nu' &= e^{i\beta\gamma_5} \nu\,, \\ u' &= e^{\ i\beta\gamma_5} \mu \qquad & \tilde{\nu}' &= e^{-i\beta\gamma_5} \tilde{\nu}\,. \end{aligned}$$

The difference between ν and $\tilde{\nu}$ is expressed as an opposite value of H. We have namely $L(\nu) = L(\tilde{\nu}) = 1$ and $H(\nu) = -\tilde{H}(\nu) = 1$.

Free Lagrangian and electromagnetic interaction

In doublet approximation of leptons we have two possibilities: $m_e = m_\mu$ and $m_e = m_\nu$. In scheme A more natural is the first possibility. We have the following expression for the free Lagrangian:

$$(3.1) \qquad \mathcal{L}_0 = \bar{L}_1 \gamma_\mu \times 1\, \partial_\mu L_1 + \bar{L}_2 \gamma_\mu \times 1\, \partial_\mu L_2 + m \bar{L}_1 1 \times \left(\frac{1+I_3}{2}\right) L_2 + h \cdot c ,$$

where $m = m_e = m_\mu$.

The mass term in (3.1) is invariant only under rotations in (1, 2) — isoplane and transformation (2.7) reduced to mass reversal $\hat{\boldsymbol{H}} = \boldsymbol{H}(\pi)$, $m \to - m$ [12]. If we assume, that the free Lagrangian is invariant under each transformation, which leaves interaction terms invariant, the continuous transformation (2.7) is reduced to the following chirality conjugation operator $\hat{\boldsymbol{H}}$:

$$(3.2) \qquad \hat{\boldsymbol{H}}: \quad \begin{array}{ll} e' = \gamma_5 e & \nu' = \gamma_5 \nu , \\ \mu' = - \gamma_5 \mu & \tilde{\nu}' = - \tilde{\gamma}_5 \nu . \end{array}$$

The electromagnetic interaction is obtained as follows:

$$(3.3) \qquad \mathcal{L}_{el} = {}^{ie}A_\mu \left\{ \bar{L}_1 \gamma_\mu \times \frac{1+I_3}{2} L_1 + \bar{L}_2 \gamma_\mu \times \frac{1+I_3}{2} L_2 \right\}$$

strictly analogous as for the $(N, \bar{\Xi})$ pair.

Leptonic interactions

Feynmann and Gell-Mann [13] have pointed out that the strangeness weak baryon current $B_\mu^{(+)}$ can be described by means of I^+. Polginhorne [14] has generalized the nucleon β-decay current to the following expression for all baryons:

$$(4.1) \qquad B_\mu^{(+)} = \bar{N}_A O_\mu \times I^+ N_A \qquad A = 1, 2, 3, 4,$$

where $O_\mu = \frac{1}{2}(1 + \gamma_5) \gamma_\mu$ and N_A — four Pais baryon doublets [15].

Now the lepton current

$$(4.2) \qquad L_\mu = e O_\mu \nu + \bar{\mu} O_\mu \nu$$

can be written as

$$(4.3) \qquad L_\mu = L_i O_\mu \times I^+ L_i \qquad (i = 1, 2)$$

and the Universal Fermi Interaction can be expressed for all strangeness conserving processes as follows

$$(4.4) \qquad H_F = g_F (J_{\mu;1} J^+_{\mu;1} + J_{\mu;2} J^+_{\mu;2}) ,$$

where

$$J_{\mu;\gamma} = \overline{N}_A O_\mu \times I_\gamma N_A + L_i O_\mu \times I_\gamma L_i .$$

The conserved weak vector current thus includes the leptonic part also.

Lippmanow [9] investigates expression (4.3) with $I^+ \to \vec{I}$. This extension is not possible in our scheme, because (3.1) is not invariant under the full 3-dimensional isogroup. Thus in this scheme for each term in L_μ we have $\Delta Q = 1$.

Scheme B

Instead of (2.3) we can obtain another scheme introducing the following two isospinors $L^1{}_{;j}$ and $L^2{}_{;j}$ $(j = 1, 2)$:

(5.1)
$$L^1: \quad L^1{}_{;1} = \begin{pmatrix} e_+ \\ \nu^c_- \end{pmatrix} \qquad L^1{}_{;2} = \begin{pmatrix} e^c_+ \\ \nu_- \end{pmatrix},$$
$$L^2: \quad L^2{}_{;1} = \begin{pmatrix} \mu_+ \\ \tilde{\nu}^c_- \end{pmatrix} \qquad L^1{}_{;2} = \begin{pmatrix} \mu^c_+ \\ \tilde{\nu}_- \end{pmatrix},$$

where $L^2{}_{;i} = \begin{pmatrix} L^i_{\alpha;j} \\ L^{i\alpha}{}_{;j} \end{pmatrix}$.

It is easily seen from (5.1) that the 4-spinors e, ν^c can be expressed with the help of $L^1{}_{;j}$ as follows:

(5.2)
$$e = \frac{1}{2}(1+\gamma_5) L^1{}_{;1} + \frac{1}{2}(1-\gamma_5) L^{1c}{}_{;2}$$
$$\nu^c = \frac{1}{2}(1-\gamma_5) L^1{}_{;1} + \frac{1}{2}(1+\gamma_5) L^{1c}{}_{;2}$$

and in a similar manner μ and $\tilde{\nu}^c$ with the help of $L^2{}_{;j}$.

(5.2) shows that in this scheme usual lepton 4-spinors have properties of Gursey's 4-spinors [16], for which E_3 — isotransformations are realized with the help of Pauli's group.

In scheme B we can express the electric charge operator Q and neutrino charge operator $\hat{N}$ with help of the following two operators

(5.3)
$$\hat{Q} = (1+\gamma_5) \times I_3 ,$$
$$\hat{N} = (1-\gamma_5) \times I_3 ,$$

where $N = Q$ (N — neutrino charge, the eigenvalue of N) only for two neutrinos.

The form (5.3) corresponds to the isorepresentation (5.1).

If we assume (1.1), the lepton number gauge transformation is

(5.4)
$$L^{i\prime} = e^{i 2\alpha \gamma_5 \times I_3} L^i ,$$

where $L^i = \begin{pmatrix} L^i{}_{;1} \\ L^i{}_{;2} \end{pmatrix}$.

It may be easily seen from (5.3) and (5.4) that for leptons we have

$$Q - N = L . \tag{5.5}$$

If we put $N \equiv Q$ for mesons and baryons, the relation (5.5) is true for all elementary particles.

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THEORETICAL PHYSICS

A New Form of the Geodesic Line Equation

by

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The equation of the geodesic line is usually written in the form:

(A)
$$\frac{d^2\xi^\alpha}{ds^2}+\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}\frac{d\xi^\beta(\xi^\beta)}{ds}\,\frac{d\xi^\gamma}{ds}=0,$$

Here the notation is obvious. As we wish to discuss the application of this equation to the General Relativity Theory, let us remark that the Greek indices run from 0 to 3 and the Latin from 1 to 3. This form is not suited to light rays but it is well suited to describe the motion of a test particle. Usually in the case of light rays, we write instead of (A):

(A′)
$$\frac{d^2\xi^\alpha}{dp^2}+\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}\frac{d\xi^\alpha}{dp}\,\frac{d\xi^\beta}{dp}=0,$$

with the additional condition:

(A″)
$$g_{\alpha\beta}\frac{d\xi^\beta}{dp}\,\frac{d\xi^\gamma}{dp}=0.$$

Here p is an arbitrary parameter but different from s because ds becomes zero for light rays. In addition we must prove that the equation does not depend on the choice of p. The parameter remains arbitrary. The proof of this theorem is not very simple.

The second drawback of (A) is the fact that for a many-body problem different ds's have to be used — as many as there are particles [1].

Thus we wish to formulate the equations of motion so as to use only one parameter for all particles and to use the same (or almost the same) reasoning for particles and light rays.

We define:

(1)
$$\dot{\xi}^\alpha=\frac{d}{dt}\,\xi^\alpha$$

that is:

(2)
$$\dot{\xi}^k=\frac{d\xi^k}{dt};\qquad \dot{\xi}^0=1,$$

and we choose $t = \xi_0$, the time, as the parameter of our motion. By μ we represent the inertial mass of the particle (1) defined by:

$$\mu^2 g_{\alpha\beta} \dot{\xi}^\alpha \dot{\xi}^\beta = m_0^2, \tag{3}$$

where m_0 is the rest mass of the particle measured in a local co-ordinate system in which:

$$\begin{gathered} \dot{\xi}^k = 0, \quad g_{\alpha\beta} = \eta^{\alpha\beta}; \quad \frac{\partial g_{\alpha\beta}}{\partial x^\varrho} = 0; \\ \eta_{oo} = 1; \quad \eta_{ok} = 0; \quad \eta_{kl} = -\delta_{kl} = \begin{cases} -1 & \text{for} \quad k = l \\ 0 & \text{for} \quad k \neq l \,. \end{cases} \end{gathered} \tag{4}$$

The transformation properties of the inertial mass μ and of the velocity $\dot{\xi}^k$ are so chosen that $m_{(0)}$ is an invariant.

We now define the equation of a geodesic line. It will be:

$$\begin{gathered} \frac{d(\mu\dot{\xi}^k)}{dt} + \mu \begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix} \dot{\xi}^\alpha \dot{\xi}^\beta = 0, \\ \mu^2 g_{\alpha\beta} \dot{\xi}^\alpha \dot{\xi}^\beta = m_0^2. \end{gathered} \tag{B}$$

These are four equations in four unknowns: $\mu, \dot{\xi}^k$. In the case of light rays we have, in place of (B):

$$\begin{gathered} \frac{d(\mu\dot{\xi}^k)}{dt} + \mu \begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix} \dot{\xi}^\alpha \dot{\xi}^\beta = 0, \\ g_{\alpha\beta} \dot{\xi}^\alpha \dot{\xi}^\beta = 0, \end{gathered} \tag{B$'$}$$

which boils down to the fact that m_0 in this case becomes zero.

We must now show that (A) is entirely equivalent to (B). Thus we must show that from (B) follows (A) and that from (A) follows (B). It can be done in this way: explicitly written, (B) is:

$$\frac{d}{dt}(\mu\dot{\xi}^k) + \mu g^{\sigma k}\left(g_{\alpha\sigma,\beta} - \frac{1}{2} g_{\alpha\beta,\sigma}\right) \dot{\xi}^\alpha \dot{\xi}^\beta = 0. \tag{5}$$

This we multiply by $\mu g_{\gamma k} \dot{\xi}^\gamma$:

$$\begin{aligned} \mu g_{k\gamma} \dot{\xi}^\gamma \frac{d}{dt}(\mu\dot{\xi}^k) + \mu^2 g^{\sigma k} g_{\gamma k}\left(g_{\alpha\sigma,\beta} - \frac{1}{2} g_{\alpha\beta,\sigma}\right) \dot{\xi}^\alpha \dot{\xi}^\beta \dot{\xi}^\gamma &= \\ = \mu g_{\alpha\gamma} \dot{\xi}^\gamma \frac{d}{dt}(\mu\dot{\xi}^\alpha) + \mu^2 g^{\sigma\varrho} g_{\gamma\varrho}\left(g_{\alpha\sigma,\beta} - \frac{1}{2} g_{\alpha\beta,\sigma}\right) \dot{\xi}^\alpha \dot{\xi}^\beta \dot{\xi}^\gamma &+ \\ - \mu g_{0\gamma} \dot{\xi}^\gamma \frac{d}{dt}(\mu) - \mu^2 g^{\sigma 0} g_{\gamma 0}\left(g_{\alpha\sigma,\beta} - \frac{1}{2} g_{\alpha\beta,\sigma}\right) \dot{\xi}^\alpha \dot{\xi}^\beta \dot{\xi}^\gamma &= \\ = \frac{1}{2} \frac{d}{dt}(g_{\alpha\beta} \dot{\xi}^\alpha \dot{\xi}^\beta \mu^2) - \mu g_{0\gamma} \dot{\xi}^\gamma \frac{d}{dt}(\mu) - \mu^2 \begin{Bmatrix} 0 \\ \sigma\beta \end{Bmatrix} \dot{\xi}^\alpha \dot{\xi}^\beta g_{0\gamma} \dot{\xi}^\gamma & \end{aligned} \tag{6}$$

But, since

$$\frac{d}{dt}(\mu^2 g_{\alpha\beta}\dot{\xi}^\alpha \dot{\xi}^\beta) = \frac{d}{dt} m_0^2 = 0, \tag{7}$$

(6) gives the following equation:

$$\frac{d}{dt}(\mu) + \mu \begin{Bmatrix} 0 \\ \beta\gamma \end{Bmatrix} \dot{\xi}^\beta \dot{\xi}^\gamma = 0. \tag{8}$$

Thus instead of (B), we have, for the equations of motion:

$$\frac{d}{dt}(\mu\dot{\xi}^\alpha) + \mu \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} \dot{\xi}^\beta \dot{\xi}^\gamma = 0. \tag{9}$$

From (9) it follows that if we introduce ds connected with μ by the relation:

$$\mu \frac{d}{dt} = \frac{d}{ds}$$

we obtain the result that (B) is identical with (A).

This is only true for particles with finite mass. For a light ray we immediately see the equivalence to (A′), choosing as a parameter

$$\mu \frac{d}{dt} = \frac{d}{dp},$$

where μ like p is, of course, arbitrary.

From what has been said it follows that we can equally well prove that (A) leads to (B) and (B) to (A). Thus we may use the equations (B) as those of a geodesic line, just as much for particles as for light rays when $m_0 = 0$. In the case of a test particle we see that the inertial mass depends on the rest mass m_0, the velocity $\dot{\xi}^k$ and on the fields $g_{\alpha\beta}$. But the motion does not depend on m_0. In this fact we see the manifestation of the equivalence relation. Thus we can replace the concept of "eigentime" with the more physical concept of inertial mass.

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БЮЛЛЕТЕНЬ ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

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Л. ЕСЬМАНОВИЧ. **О РАЗЛОЖЕНИЯХ АБЕЛЕВЫХ БЕЗТОРСИОННЫХ ГРУПП НА ПРОСТЫЕ СЛАГАЕМЫЕ** . . . . стр. 505—510

В 1945 году Б. Джонсон построил абелевую безторсионную четырехмерную группу, обладающую двумя разными разложениями на простую сумму двух неразложимых слагаемых. В 1958 году Л. Фукс в книге „Abelian Groups" выдвинул следующую проблему:

Пусть n — произвольное натуральное число больше 4 и пусть даны два разных разложения этого числа на сумму двух натуральных чисел $n = r_1 + r_2 = s_1 + s_2$. Существует ли безторсионная абелева n-мерная группа, обладающая двумя разными разложениями на простую сумму двух неразложимых слагаемых и таких, что числа r_1, r_2 являются соответственно размерностями слагаемых первого разложения, а числа s_1, s_2 — размерностями слагаемых второго разложения?

В работе автор дает не только положительное решение проблемы Фукса, но также метод построения безторсионных абелевых n-мерных групп, обладающих двумя разными разложениями на простую сумму трех неразложимых слагаемых с заранее заданными размерами. Этот метод состоит в нахождении некоторых целочисленных матриц третьего порядка с определением ± 1, элементы которых выполняют ряд условий делимости.

Ч. РЫЛЛЬ-НАРДЗЕВСКИЙ, **НЕКОТОРЫЙ АНАЛОГ ТЕОРЕМЫ ФУБИНИ И ЕГО ПРИЛОЖЕНИЯ К СЛУЧАЙНЫМ ЛИНЕЙНЫМ УРАВНЕНИЯМ.** стр. 511—513

Пусть $(\Omega, \mathcal{M}, \mu)$ пространство с σ-конечной мерой μ, определенной на σ-теле $\mathcal{M}$ подмножеств множества M; пусть $(\Lambda, \mathcal{B})$ — борелевское пространство, т. е. такое абстрактное множество Λ с отмеченным σ-телом $\mathcal{B}$ его подмножеств, что его можно взаимно однозначно отобразить на некоторое борелевское подмножество полного, сепарабельного, метрического пространства таким образом, чтобы B перешло на класс всех борелевских подмножеств образа.

Пусть Z измеримое подмножество $\Omega \times \Lambda$. Пусть $Z_\omega = \{\lambda : (\omega, \lambda) \in Z\}$ и $Z_\lambda = \{\omega : (\omega, \lambda) \in Z$.

Доказывается теорема 1: если множество Z_ω счётно для почти всех ω, то $\mu(Z_\lambda) = 0$ для всех λ, за исключением множества значений мощности не более, чем $\aleph_0$.

Приводится пример, показывающий, что относительно $(\Lambda, \mathcal{B})$ предположения существенны.

Как следствие из теоремы 1 получается теорема 2, относящаяся к теории линейных уравнений, зависящих от случайного параметра в пространствах Банаха.

Ю. МЕДЕР, **О СИЛЬНОЙ СУММИРУЕМОСТИ ОРТОГОНАЛЬНЫХ РЯДОВ.**
стр. 515—517

Определение I.

Ряд

$$A_1 + A_2 + \dots + A_n + \dots \tag{1}$$

имеющий n-тые частичные суммы s_n, именуемый, при данной возрастающей последовательности указателей $\{\nu_k\}$ суммируется $(R^{(\nu)}, 1)$ к s, когда

$$\tau_n^{(\nu)} = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} s_{\nu_k} \to s, \qquad n \to \infty.$$

В частности, когда $\nu_k \equiv k$, ряд (1) суммируется $(R, 1)$ к s.

Определение II.

Ряд (1), при данной возрастающей последовательности указателей $\{\nu_k\}$, именуемый сильно суммируемым $(R_p^{(\nu)}, 1)$ с показателем p к s, когда

$$\sum_{k=1}^{n} \frac{1}{k} |s_{\nu_k} - s|^p = 0 (\log n), \qquad n \to \infty.$$

В частном случае, когда $\nu_k = k$, $p = 1$, тогда ряд (1) сильно суммируется $(R, 1)$ к s.

Предположим, что $ON\{\varphi_n(x)\}$ обозначает ортонормированную систему, определенную в интервале $\langle 0, 1 \rangle$ и

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \tag{2}$$

есть ортогональный ряд, в котором коэффициенты $\{c_n\} \in l^2$ т. е.

$$\sum_{n=1}^{\infty} c_n^2 < \infty. \tag{3}$$

Теорема 1. *Пусть* $\{c_n^*\} \epsilon l^2$ *является последовательностью вещественных, положительных чисел, удовлетворяющих неравенству*

$$\sqrt{n \log n}\; c_n^* \geqslant \sqrt{(n+1)\log(n+1)}\; c_{n+1}^* \; (n=2,3,\dots)$$

а $\{c_n\}$ — *любой последовательностью вещественных чисел, для которых*

$$c_n = 0\,(c_n^*).$$

Если ряд (2) *составленный из этих коэффициентов почти везде в* $\langle 0,1\rangle$ *суммируется* $(R,1)$ *до некоторой функции* $f(x)$, *тогда этот ряд почти везде сильно суммируется* $(R_2^{(\nu)},1)$ *к* $f(x)$ *для всякой возрастающей последовательности указателей* $\{\nu_k\}$.

Теорема 2. *Существует такая система* $ON\{\Phi_n(x)\}$, *последовательность коэффициентов* $\{c_n \epsilon l^2\}$ *и последовательность указателей* $\{\nu_k\}$, *что ряд* $\sum\limits_{n=1}^{\infty} c_n \Phi_n(x)$ *почти везде в* $\langle 0,1\rangle$ *суммируется* $(R,1)$ *до некоторой функции* $f(x)$, *но почти не везде в* $\langle 0,1\rangle$ *сильно суммируется.*

Теорема 3. *Если*

$$\sum_{n=8}^{\infty} c_n^2 (\log\log\log n)^2 < \infty,$$

тогда существует такая функция $f(x) \epsilon L^2$, *что ряд* (2) *почти везде в* $\langle 0,1\rangle$ *сильно суммируется* $(R_2^{(\nu)},1)$ *к* $f(x)$ *для всякой возрастающей последовательности указателей* $\{\nu_k\}$.

М. Маховер, **ЗАМЕТКА О ПРЕДЛОЖЕНИЯХ, СОХРАНЯЮЩИХ СПРАВЕДЛИВОСТЬ ПРИ ОПЕРАЦИЯХ ПРОСТЫХ ПРОИЗВЕДЕНИЙ И ВОЗВЕДЕНИЯ В СТЕПЕНЬ** . . . . стр. 519—523

В работе приводятся результаты, касающиеся классов предложений (в языке, являющемся частью предикатного исчисления первого порядка с равенством), которые сохраняют справедливость при операциях простых произведений и возведения в степень произвольной или данной (конечной или бесконечной) „длины".

Доказывается, что все эти классы рекурсивно перечислимы. Если данный язык содержит только такие предикатные буквы, которые имеют одно пустое место, то рассматриваемые классы рекурсивны; во всех других случаях (за исключением случая, когда язык содержит лишь одну предикатную букву, причем эта буква имеет два пустых места — этот вопрос остается не решенным) ни один из этих классов не является рекурсивным.

В. Жаковский, **СВОЙСТВА ОБОБЩЕННОГО СИНГУЛЯРНОГО ИНТЕГРАЛА ТИПА КОШИ** . . . . стр. 525—529

Автор определяет обобщенный сингулярный интеграл типа Коши формулой (3). Автор доказывает обобщенную формулу Пуанкарэ-Бертрана (11), принимая интеграл Коши в обобщенном смысле и выступа-

ющие функции класса $\mathfrak{H}_{\alpha}^{\mu}$, определенные на совокупности L разомкнутых дуг и контуров. Кроме того, автор доказывает формулы (15), аналогичные формулам Племеля для интеграла типа Коши в обыкновенном смысле главного значения.

И. МУСЕЛЯК и В. ОРЛИЧ, **ОБОБЩЕНИЕ НЕКОТОРЫХ ТЕОРЕМ О РАСШИРЕНИИ** . . . . , ... стр. 531—534

Авторы рассматривают теорему о расширении аддитивных однородных функционалов и теорему о неравенствах (сравн. [1]), обобщая их для случая когда майоранта является выпуклым функционалом, который может принимать бесконечное значение, вместо функционала Банаха. Полученный результат содержит теорему о расширении Г. Наканы (сравн. [2]).

А. ДЫНИН и Б. МИТЯГИН, **КРИТЕРИЙ ЯДЕРНОСТИ В ТЕРМИНАХ АППРОКСИМАТИВНОЙ РАЗМЕРНОСТИ** . . . . стр. 535—540

Для любого множества A и произвольной окрестности нуля U в локально выпуклом пространстве E обозначим через $N(A,U;\varepsilon)$ число элементов минимального εU-покрытия множества A. Порядком $\varrho(U,U)$ множества A относительно U называется $\overline{\lim\limits_{\varepsilon\to 0}}\,\{\log N(A,U;\varepsilon)/\log 1/\varepsilon\}$.

В ядерном пространстве для каждого ограниченного множества A и любой окрестности U справедливо соотношение $\varrho\,(A,U)=0$ (теорема 1). Это условие и является достаточным для ядерности многих метризуемых (и их сильных сопряженных) пространств (теорема 2). При доказательстве теоремы 2 основную роль играют леммы 2 и 2а Б. Митягина, в которых оцениваются порядки некоторых множеств в l^p и в нормированных пространствах с базисом Шаудера.

Использованная в работе методика позволяет доказать, что (теорема 3) *всякий базис Шаудера в ядерном пространстве Фреше является безусловным базисом.*

Б. КАРЧЕВСКИЙ, **ПОЛЯРИЗАЦИЯ ИЗОГНУТОЙ ЭЛЕКТРОМАГНИТНОЙ ВОЛНЫ** . . . . стр. 541—546

В работе исследована поляризация изогнутой волны в случае дифракции Фраунгофера, принимая теорию Каттлера как исходную точку.

Оказалось, что поляризация волны изогнутой под произвольным углом является такой же, как и поляризация падающей волны.

Г. ЦОФТА, **ВОЗМОЖНОСТЬ НОВОГО ЯВЛЕНИЯ В АНТИФЕРРОМАГНЕТИКАХ** . . . . стр. 547—551

Автор представляет результаты теоретических исследований, касающихся сверхструктурной анизотропии дисперсии спиновых волн. В антиферромагнетиках чисто геометрические исследования позволя-

ют предвидеть, при каких обстоятельствах выступит это явление и какой характер оно будет иметь. Теоретические подсчеты подтверждают эти гипотезы во всех случаях.

Показано, что измерения сверхструктурной анизотропии дисперсии позволяют определить отношение $\eta = J_1/J_2$ обменного интеграла для взаимодействия между ближайшими соседями (J_1) и следующими за ними (J_2). Автором предложены два метода измерений.

Г. ЛЮКЕРСКИЙ, **ПРЕДСТАВЛЕНИЯ ЛЕПТОНОВ В ИЗОТОПИЧЕСКОМ ПРОСТРАНСТВЕ** . . . . стр. 553—557

Лептоны описываются двумя изотопическими дублетами. Полагаем для лептонов $S=0$. Четвертым лептоном (кроме $e_{\pm}$, μ и ν) является второе нейтрино $\tilde{\nu}$. Эти нейтрино ν и $\tilde{\nu}$ отличаются зарядом H, который получается в результате обобщения преобразования киральности $\psi \to \pm\gamma_5\psi$.

Строятся две классификационные схемы лептонов A и B. Для первой из них построены: лагранжиан свободного поля, взаимодействия с электромагнитным полем и лагранжиан слабого взаимодействия.

Предлагаемые представления строятся в изотопическом пространстве E_3. После введения представлений группы $\mathbf{C}'$, получено дополнение классификации элементарных частиц Жевуского [11] в изопространстве Минковского .

Л. ИНФЕЛЬД, **НОВЫЙ ВИД УРАВНЕНИЯ ГЕОДЕЗИЧЕСКОЙ ЛИНИИ** стр. 559—561

В работе автор сугеррирует новый вид уравнения геодезической линии, применимый так для частиц (молекул), как и для световых лучей, а именно:

$$\mu^2 g_{\alpha\beta}\dot{\xi}^\alpha\dot{\xi}^\beta = m_{(0)}^2$$

$$\frac{d(\mu\dot{\xi}^k)}{dt} + \mu\left\{ {k \atop \alpha\beta} \right\}\dot{\xi}^\alpha\dot{\xi}^\beta = 0\,,$$

где

$$\dot{\xi}^k = \frac{d\xi^k}{dt}; \qquad \dot{\xi}^0 = 1;$$

латинские индексы обозначены от 1 до 3, а греческие от 0 до 3.

Для световых лучей $m_{(0)} = 0$.

TABLE DES MATIÈRES

Mathématique

Physique Théorique

Cena zł 20.—